Basic Introduction to Nonlinear Analysis

Ronald D. Ziemian
Bucknell University

NASCC THE STEEL CONFERENCE incorporating the 2007 Annual Stability Conference
NEW ORLEANS Ernest N. Morial Convention Center
Exhibits April 18-20, 2007 • Conference April 18-21, 2007

The function of a structural engineer is to design — not to analyze

Norris and Wilbur
1960

Analysis is a means to an end rather than the end itself.

Role of the analysis:

- forces, moments and deflections ⇒ design equations
- insight into the behavior of a structure
  ⇒ better the understanding, better the design

Limit States Design:

Prior to limit of resistance, significant nonlinear response, including
- geometrical effects (P-Δ, P-δ)
- material effects (yielding, cracking, crushing)
- combined effects

Impetus:

Limit States Design

AISC Ch. C: P-Δ, P-δ (App. 7)
App. 1: Inelastic Design
Seismic: Pushover Analysis
Other: Progressive Collapse

Nonlinear Analysis

Available Software
Research
Education

Nonlinear Analysis

- Hand methods
  - Second-order effects
    - i.e. Moment Amplification Factors (B1 and B2 factors)
  - Material nonlinear effects
    - i.e. plastic analysis (upper and lower bound theories)
- Computer Methods (focus of today's lecture)
  - Lots of variations
    - all use same basic concepts (most important to today)
    - one approach will be presented (basis for MASTAN2)
- Please keep in mind
  - All methods are approximate
  - Not a substitute, but a complement to good engineering

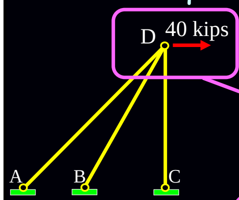
Lecture Overview

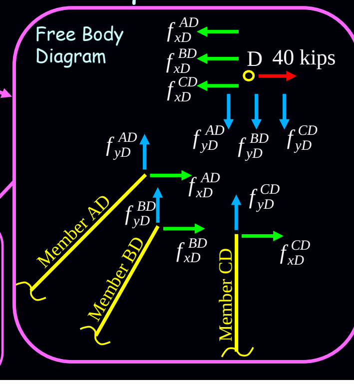
- Brief Introduction (done!)
- Computer Structural Analysis (Review?)
- Basis for Material Nonlinear Models
- Incorporating Geometric Nonlinear Behavior
- Critical Load Analysis
- Overview of MASTAN2 software
- Summary and Concluding Remarks

How does the computer get these results?

- State-of-the-Art Crystal Ball? Not quite.
- By applying 2 requirements and 1 translator
  - Two Requirements:
    - Equilibrium (equations in terms of F's and M's, 1 per d.o.f.)
    - Compatibility (equations in terms of Δ's and θ's, 1 per d.o.f.)
  - Translator "apples to oranges"
    - Constitutive Relationship (i.e. Hooke's Law, σ = E ϵ)
    - Generalized to Force-to-Displacement (i.e. F=kΔ)
    - Re-write equilibrium eqs. in terms of unknown displacements
- # of Equil. Eqs. = # of Unknown Displacements

Equilibrium Equations





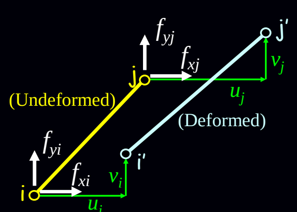
x-d.o.f. u_D: $\sum F_x = 0$

$$40 = f_{xD}^{AD} + f_{xD}^{BD} + f_{xD}^{CD}$$

y-d.o.f. v_D: $\sum F_y = 0$

$$0 = f_{yD}^{AD} + f_{yD}^{BD} + f_{yD}^{CD}$$

Translator: Forces → Displacements

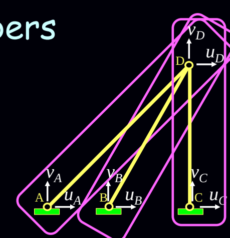


$$f_{xi} = k_{11}u_i + k_{12}v_i + k_{13}u_j + k_{14}v_j$$
$$f_{yi} = k_{21}u_i + k_{22}v_i + k_{23}u_j + k_{24}v_j$$
$$f_{xj} = k_{31}u_i + k_{32}v_i + k_{33}u_j + k_{34}v_j$$
$$f_{yj} = k_{41}u_i + k_{42}v_i + k_{43}u_j + k_{44}v_j$$

Big Question:
Where do these known stiffness coefficients k's come from?

Little Answer:
Function of member's material and geometric properties, including its orientation.

F → Δ for all members



Member AD:

$$f_{xA}^{AD} = k_{11}^{AD}u_A^{AD} + k_{12}^{AD}v_A^{AD} + k_{13}^{AD}u_D^{AD} + k_{14}^{AD}v_D^{AD}$$
$$f_{yA}^{AD} = k_{21}^{AD}u_A^{AD} + k_{22}^{AD}v_A^{AD} + k_{23}^{AD}u_D^{AD} + k_{24}^{AD}v_D^{AD}$$
$$f_{xD}^{AD} = k_{31}^{AD}u_A^{AD} + k_{32}^{AD}v_A^{AD} + k_{33}^{AD}u_D^{AD} + k_{34}^{AD}v_D^{AD}$$
$$f_{yD}^{AD} = k_{41}^{AD}u_A^{AD} + k_{42}^{AD}v_A^{AD} + k_{43}^{AD}u_D^{AD} + k_{44}^{AD}v_D^{AD}$$

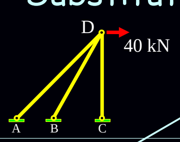
Member BD:

$$f_{xB}^{BD} = k_{11}^{BD}u_B^{BD} + k_{12}^{BD}v_B^{BD} + k_{13}^{BD}u_D^{BD} + k_{14}^{BD}v_D^{BD}$$
$$f_{yB}^{BD} = k_{21}^{BD}u_B^{BD} + k_{22}^{BD}v_B^{BD} + k_{23}^{BD}u_D^{BD} + k_{24}^{BD}v_D^{BD}$$
$$f_{xD}^{BD} = k_{31}^{BD}u_B^{BD} + k_{32}^{BD}v_B^{BD} + k_{33}^{BD}u_D^{BD} + k_{34}^{BD}v_D^{BD}$$
$$f_{yD}^{BD} = k_{41}^{BD}u_B^{BD} + k_{42}^{BD}v_B^{BD} + k_{43}^{BD}u_D^{BD} + k_{44}^{BD}v_D^{BD}$$

Member CD:

$$f_{xC}^{CD} = k_{11}^{CD}u_C^{CD} + k_{12}^{CD}v_C^{CD} + k_{13}^{CD}u_D^{CD} + k_{14}^{CD}v_D^{CD}$$
$$f_{yC}^{CD} = k_{21}^{CD}u_C^{CD} + k_{22}^{CD}v_C^{CD} + k_{23}^{CD}u_D^{CD} + k_{24}^{CD}v_D^{CD}$$
$$f_{xD}^{CD} = k_{31}^{CD}u_C^{CD} + k_{32}^{CD}v_C^{CD} + k_{33}^{CD}u_D^{CD} + k_{34}^{CD}v_D^{CD}$$
$$f_{yD}^{CD} = k_{41}^{CD}u_C^{CD} + k_{42}^{CD}v_C^{CD} + k_{43}^{CD}u_D^{CD} + k_{44}^{CD}v_D^{CD}$$

Substituting into Equil. Eqs.



Member AD:
$$f_{xD}^{AD} = k_{31}^{AD}u_A^{AD} + k_{32}^{AD}v_A^{AD} + k_{33}^{AD}u_D^{AD} + k_{34}^{AD}v_D^{AD}$$

Member BD:
$$f_{xD}^{BD} = k_{31}^{BD}u_B^{BD} + k_{32}^{BD}v_B^{BD} + k_{33}^{BD}u_D^{BD} + k_{34}^{BD}v_D^{BD}$$

Member CD:
$$f_{xD}^{CD} = k_{31}^{CD}u_C^{CD} + k_{32}^{CD}v_C^{CD} + k_{33}^{CD}u_D^{CD} + k_{34}^{CD}v_D^{CD}$$

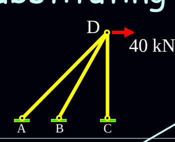
x-d.o.f. u_D: $\sum F_x = 0$

$$40 = f_{xD}^{AD} + f_{xD}^{BD} + f_{xD}^{CD}$$

d.o.f. u_D:

$$40 = \left(k_{31}^{AD}u_A^{AD} + k_{32}^{AD}v_A^{AD} + k_{33}^{AD}u_D^{AD} + k_{34}^{AD}v_D^{AD}\right) + \left(k_{31}^{BD}u_B^{BD} + k_{32}^{BD}v_B^{BD} + k_{33}^{BD}u_D^{BD} + k_{34}^{BD}v_D^{BD}\right) + \left(k_{31}^{CD}u_C^{CD} + k_{32}^{CD}v_C^{CD} + k_{33}^{CD}u_D^{CD} + k_{34}^{CD}v_D^{CD}\right)$$

Substituting into Equil. Eqs. (cont.)



Member AD:
$$f_{yD}^{AD} = k_{41}^{AD}u_A^{AD} + k_{42}^{AD}v_A^{AD} + k_{43}^{AD}u_D^{AD} + k_{44}^{AD}v_D^{AD}$$

Member BD:
$$f_{yD}^{BD} = k_{41}^{BD}u_B^{BD} + k_{42}^{BD}v_B^{BD} + k_{43}^{BD}u_D^{BD} + k_{44}^{BD}v_D^{BD}$$

Member CD:
$$f_{yD}^{CD} = k_{41}^{CD}u_C^{CD} + k_{42}^{CD}v_C^{CD} + k_{43}^{CD}u_D^{CD} + k_{44}^{CD}v_D^{CD}$$

y-d.o.f. v_D: $\sum F_y = 0$

$$0 = f_{yD}^{AD} + f_{yD}^{BD} + f_{yD}^{CD}$$

d.o.f. v_D:

$$0 = \left(k_{41}^{AD}u_A^{AD} + k_{42}^{AD}v_A^{AD} + k_{43}^{AD}u_D^{AD} + k_{44}^{AD}v_D^{AD}\right) + \left(k_{41}^{BD}u_B^{BD} + k_{42}^{BD}v_B^{BD} + k_{43}^{BD}u_D^{BD} + k_{44}^{BD}v_D^{BD}\right) + \left(k_{41}^{CD}u_C^{CD} + k_{42}^{CD}v_C^{CD} + k_{43}^{CD}u_D^{CD} + k_{44}^{CD}v_D^{CD}\right)$$

So, where are we at?

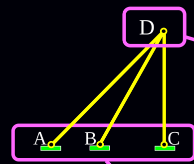
- We have two equilibrium equations (1 per d.o.f.) in terms of a lot of displacements:

$$u_D: \quad 40 = \left(k_{31}^{AD}u_A^{AD} + k_{32}^{AD}v_A^{AD} + k_{33}^{AD}u_D^{AD} + k_{34}^{AD}v_D^{AD}\right) + \left(k_{31}^{BD}u_B^{BD} + k_{32}^{BD}v_B^{BD} + k_{33}^{BD}u_D^{BD} + k_{34}^{BD}v_D^{BD}\right) + \left(k_{31}^{CD}u_C^{CD} + k_{32}^{CD}v_C^{CD} + k_{33}^{CD}u_D^{CD} + k_{34}^{CD}v_D^{CD}\right)$$

$$v_D: \quad 0 = \left(k_{41}^{AD}u_A^{AD} + k_{42}^{AD}v_A^{AD} + k_{43}^{AD}u_D^{AD} + k_{44}^{AD}v_D^{AD}\right) + \left(k_{41}^{BD}u_B^{BD} + k_{42}^{BD}v_B^{BD} + k_{43}^{BD}u_D^{BD} + k_{44}^{BD}v_D^{BD}\right) + \left(k_{41}^{CD}u_C^{CD} + k_{42}^{CD}v_C^{CD} + k_{43}^{CD}u_D^{CD} + k_{44}^{CD}v_D^{CD}\right)$$

What card haven't we played yet?

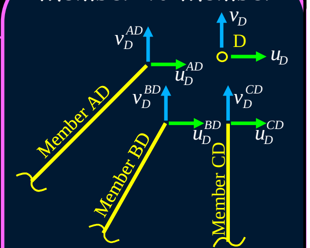
Compatibility Eqs. (consistent deflections)



Member-to-Support

$$u_A = u_A^{AD} = 0 \qquad u_C = u_C^{CD} = 0$$
$$v_A = v_A^{AD} = 0 \qquad v_C = v_C^{CD} = 0$$
$$u_B = u_B^{BD} = 0$$
$$v_B = v_B^{BD} = 0$$

Member-to-Member



$$u_D = u_D^{AD} = u_D^{BD} = u_D^{CD}$$
$$v_D = v_D^{AD} = v_D^{BD} = v_D^{CD}$$

Time for some serious simplifying

- Applying Compatibility to Equil. Eqs.:

$$u_D: \quad 40 = \left(k_{31}^{AD}u_A^{AD} + k_{32}^{AD}v_A^{AD} + k_{33}^{AD}u_D^{AD} + k_{34}^{AD}v_D^{AD}\right) + \left(k_{31}^{BD}u_B^{BD} + k_{32}^{BD}v_B^{BD} + k_{33}^{BD}u_D^{BD} + k_{34}^{BD}v_D^{BD}\right) + \left(k_{31}^{CD}u_C^{CD} + k_{32}^{CD}v_C^{CD} + k_{33}^{CD}u_D^{CD} + k_{34}^{CD}v_D^{CD}\right)$$

$$v_D: \quad 0 = \left(k_{41}^{AD}u_A^{AD} + k_{42}^{AD}v_A^{AD} + k_{43}^{AD}u_D^{AD} + k_{44}^{AD}v_D^{AD}\right) + \left(k_{41}^{BD}u_B^{BD} + k_{42}^{BD}v_B^{BD} + k_{43}^{BD}u_D^{BD} + k_{44}^{BD}v_D^{BD}\right) + \left(k_{41}^{CD}u_C^{CD} + k_{42}^{CD}v_C^{CD} + k_{43}^{CD}u_D^{CD} + k_{44}^{CD}v_D^{CD}\right)$$

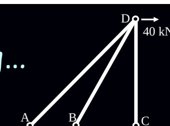
All = u_D

All = 0

All = v_D

Which simplifies to...

After simplifying...



$$u_D: \quad 40 = \left(k_{33}^{AD} + k_{33}^{BD} + k_{33}^{CD}\right)u_D^{?} + \left(k_{34}^{AD} + k_{34}^{BD} + k_{34}^{CD}\right)v_D^{?}$$

$$v_D: \quad 0 = \left(k_{43}^{AD} + k_{43}^{BD} + k_{43}^{CD}\right)u_D^{?} + \left(k_{44}^{AD} + k_{44}^{BD} + k_{44}^{CD}\right)v_D^{?}$$

Since k's are known, we have 2 Equations and 2 Unknowns

Solve for Unknown Displacements

$u_D = \#$ and $v_D = \#\#$

With all displacements, solve for member forces...

$$u_A = u_A^{AD} = 0 \qquad v_A = v_A^{AD} = 0$$
$$u_B = u_B^{BD} = 0 \qquad v_B = v_B^{BD} = 0$$
$$u_C = u_C^{CD} = 0 \qquad v_C = v_C^{CD} = 0$$
$$u_D = u_D^{AD} = u_D^{BD} = u_D^{CD} = \#$$
$$v_D = v_D^{AD} = v_D^{BD} = v_D^{CD} = \#\#$$

Member AD:

$$f_{xA}^{AD} = k_{11}^{AD}u_A^{AD} + k_{12}^{AD}v_A^{AD} + k_{13}^{AD}u_D^{AD} + k_{14}^{AD}v_D^{AD}$$
$$f_{yA}^{AD} = k_{21}^{AD}u_A^{AD} + k_{22}^{AD}v_A^{AD} + k_{23}^{AD}u_D^{AD} + k_{24}^{AD}v_D^{AD}$$
$$f_{xD}^{AD} = k_{31}^{AD}u_A^{AD} + k_{32}^{AD}v_A^{AD} + k_{33}^{AD}u_D^{AD} + k_{34}^{AD}v_D^{AD}$$
$$f_{yD}^{AD} = k_{41}^{AD}u_A^{AD} + k_{42}^{AD}v_A^{AD} + k_{43}^{AD}u_D^{AD} + k_{44}^{AD}v_D^{AD}$$

Member BD:

$$f_{xB}^{BD} = k_{11}^{BD}u_B^{BD} + k_{12}^{BD}v_B^{BD} + k_{13}^{BD}u_D^{BD} + k_{14}^{BD}v_D^{BD}$$
$$f_{yB}^{BD} = k_{21}^{BD}u_B^{BD} + k_{22}^{BD}v_B^{BD} + k_{23}^{BD}u_D^{BD} + k_{24}^{BD}v_D^{BD}$$
$$f_{xD}^{BD} = k_{31}^{BD}u_B^{BD} + k_{32}^{BD}v_B^{BD} + k_{33}^{BD}u_D^{BD} + k_{34}^{BD}v_D^{BD}$$
$$f_{yD}^{BD} = k_{41}^{BD}u_B^{BD} + k_{42}^{BD}v_B^{BD} + k_{43}^{BD}u_D^{BD} + k_{44}^{BD}v_D^{BD}$$

Member CD:

$$f_{xC}^{CD} = k_{11}^{CD}u_C^{CD} + k_{12}^{CD}v_C^{CD} + k_{13}^{CD}u_D^{CD} + k_{14}^{CD}v_D^{CD}$$
$$f_{yC}^{CD} = k_{21}^{CD}u_C^{CD} + k_{22}^{CD}v_C^{CD} + k_{23}^{CD}u_D^{CD} + k_{24}^{CD}v_D^{CD}$$
$$f_{xD}^{CD} = k_{31}^{CD}u_C^{CD} + k_{32}^{CD}v_C^{CD} + k_{33}^{CD}u_D^{CD} + k_{34}^{CD}v_D^{CD}$$
$$f_{yD}^{CD} = k_{41}^{CD}u_C^{CD} + k_{42}^{CD}v_C^{CD} + k_{43}^{CD}u_D^{CD} + k_{44}^{CD}v_D^{CD}$$

Summary of Computer Approach

- For each d.o.f., write an equilibrium equation:

  $F_{external} = \Sigma f_{member}$

- Re-write (translate) each member force in terms of its end displacements (Stiffness Eqs.)

  $f_{member} = \Sigma k_{member} \Delta_{member\ end}$

- Substitute Stiffness Eqs. into above Equil. Eqs.
- Simplify Equil. Eqs. by applying member-to-member and member-to-support compatibility conditions
- Solve n Equil. Eqs. for the n unknown displacements
- Use Stiffness Eqs. to calculate member forces
- Apply Equil. Eqs. to solve for reactions

Lot's of Questions

- So, this is how most commercial programs such as SAP2000, RISA, STAAD, etc. get the answer?
  - Yes! Known as "Direct Stiffness Method"
- So, all such programs will give the same answer?
  - Yes, as long as it is a static 1st-order elastic analysis.
- Wait a minute…Is this the basic analysis procedure for the "finite element method"?
  - Yes! Bit more tricky to get k's, σ's, and ε's

Two Big Questions

- Where do those stiffness coefficients come from?
  - You mean the ones that relate member end forces to member end displacements?
  - Yeah, those k's ! <More to come on this>
- What happens when we go static nonlinear or even dynamic?
  - Same basic procedure, but apply loads in increments and perform a series of analyses. Then, sum incremental results.

‹ Much more to come on this! ›

Important Points

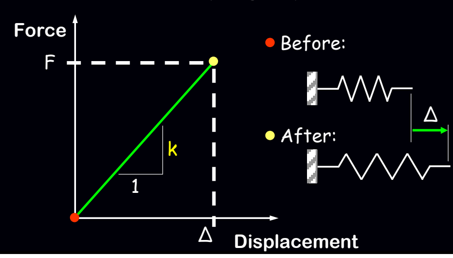
- The only opportunity for most computer analysis software to model the actual behavior of the structure is through the member stiffness terms.
- So, to include
  - first-order effects
  - second-order effects
  - material nonlinear behavior

Must modify member stiffness!!!

- Let's review member stiffness

Stiffness Coefficients, k's

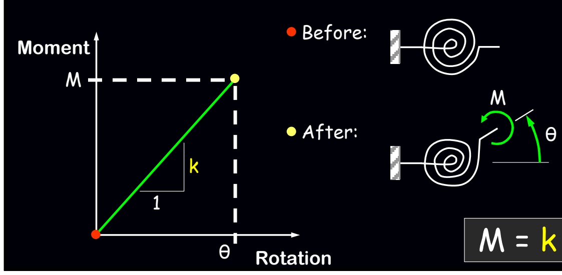
- Let's start with high school physics
  - Extension Spring Experiment



$F = k\,\Delta$

Stiffness Coefficients, k's (cont.)

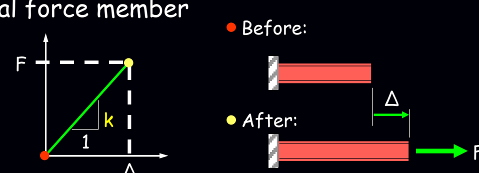
- More "advanced" high school physics lab
  - Rotational Spring Experiment



$M = k\,\theta$

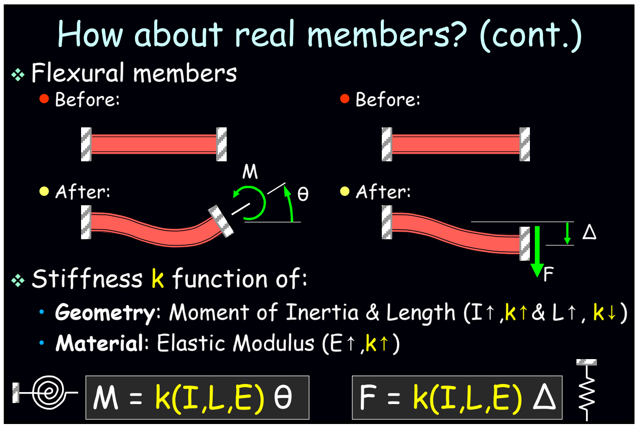
How about real structural members?

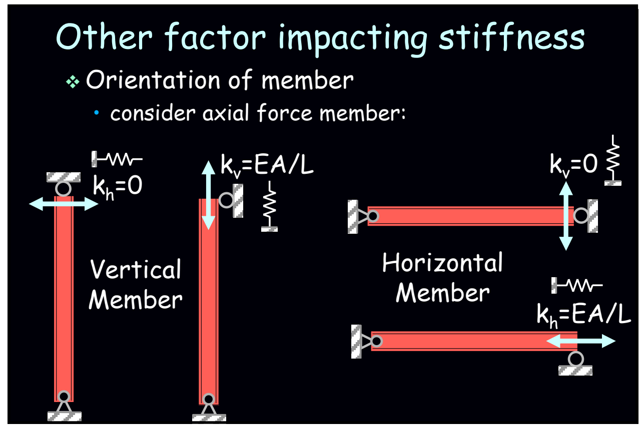
- Axial force member

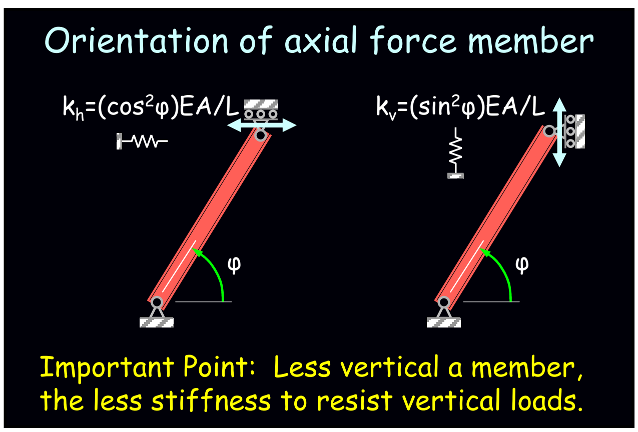


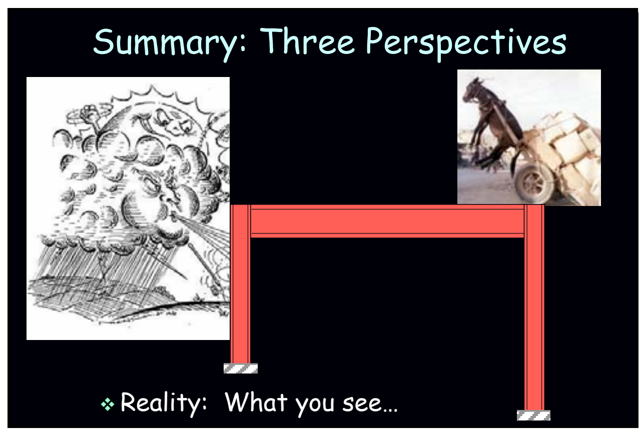
- Stiffness k function of:
  - **Geometry**: Area and Length (A↑,k↑ & L↑, k↓)
  - **Material**: Elastic Modulus (E↑,k↑)

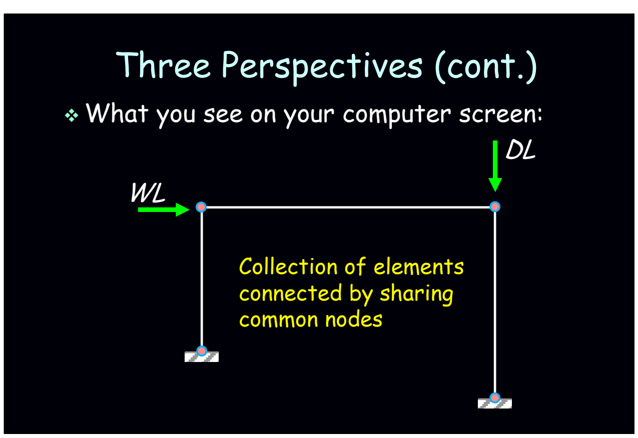
$F = k(A,L,E)\,\Delta$

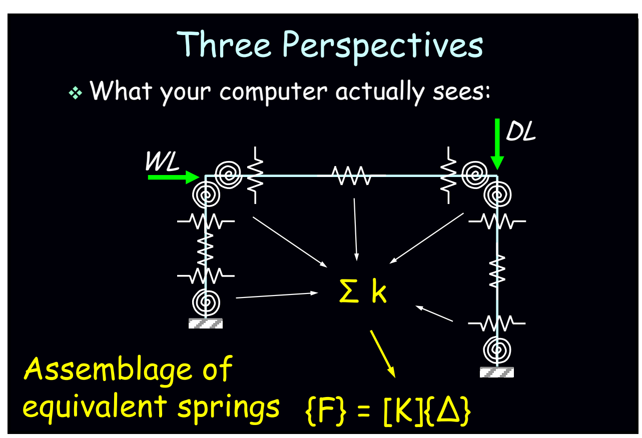

How about real members? (cont.)
Flexural members
Before:
After:
M
θ
Before:
After:
Δ
F
Stiffness k function of:
Geometry: Moment of Inertia & Length (I↑,k↑& L↑, k↓)
Material: Elastic Modulus (E↑,k↑)
M = k(I,L,E) θ
F = k(I,L,E) Δ


Other factor impacting stiffness
Orientation of member
consider axial force member:
k_h=0
k_v=EA/L
Vertical Member
k_v=0
Horizontal Member
k_h=EA/L


Orientation of axial force member
$k_h=(\cos^2\varphi)EA/L$
$k_v=(\sin^2\varphi)EA/L$
φ
φ
Important Point: Less vertical a member, the less stiffness to resist vertical loads.


Summary: Three Perspectives
Reality: What you see…


Three Perspectives (cont.)
What you see on your computer screen:
DL
WL
Collection of elements connected by sharing common nodes

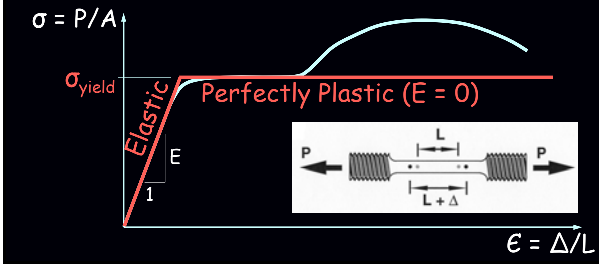

Three Perspectives
What your computer actually sees:
DL
WL
Σ k
Assemblage of equivalent springs
{F} = [K]{Δ}

Analysis Review: Key Points

- Reviewed the "Direct Stiffness Method"
  - Equilibrium → Translator F(Δ) → Compatibility
- Response of structure controlled by stiffness of members (a.k.a. springs)
- First-order elastic stiffness of member function of:
  - Material Property (E)
  - Geometric Properties (A, I, L, and orientation)
- Time to go nonlinear… let's begin with material nonlinear

Material Nonlinear (Inelastic)

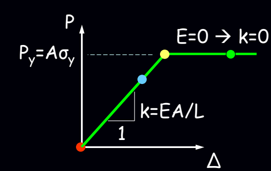
- Best place to start is with a tensile test



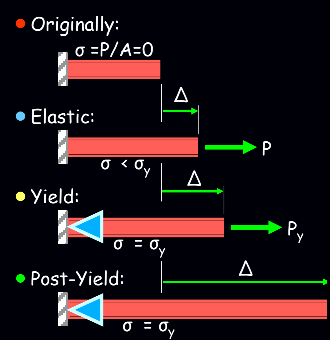
Normal Stress: Structural Members

- For typical structural steel members (L/d>10), elastic/inelastic behavior controlled by normal stresses σ's acting along the length axis of the member.
- Normal stress produced by:
  - Axial force (P/A)
  - Major and/or minor axis flexure (Mc/I)
  - Combination of above effects (i.e. P/A + Mc/I)
  - Warping (not today!)
- We will assume elastic-perfectly-plastic material (often done for steel)

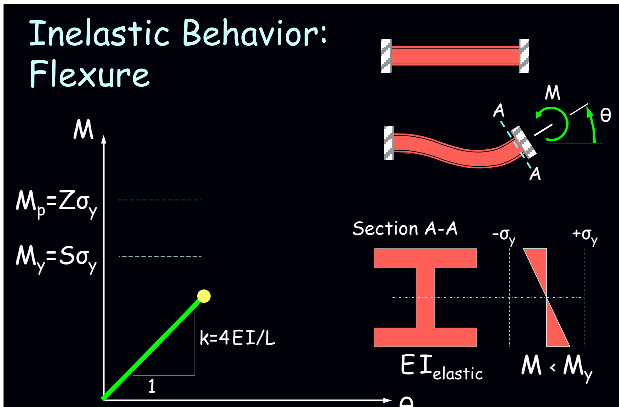
Inelastic Behavior: Axial Force



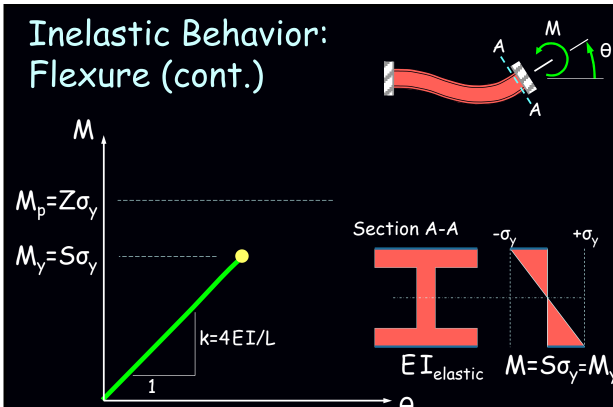
Plastic Hinge at $P = P_y$ or when $P/P_y = 1.0$

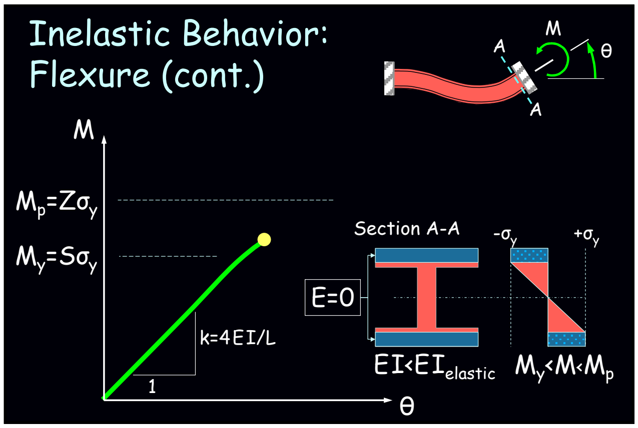


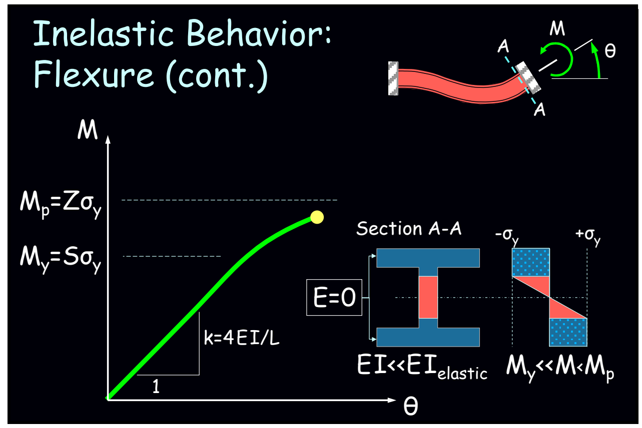
Inelastic Behavior: Flexure

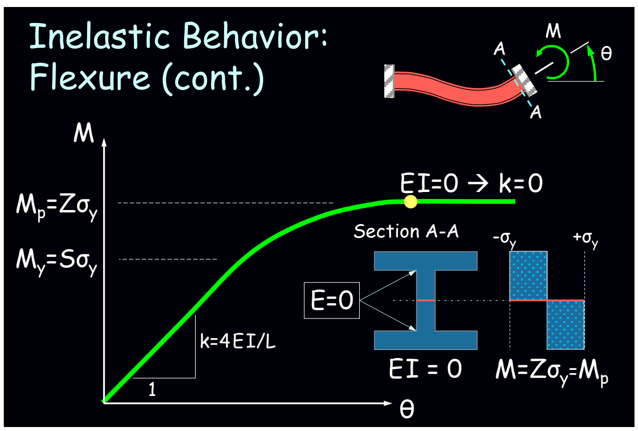


Inelastic Behavior: Flexure (cont.)

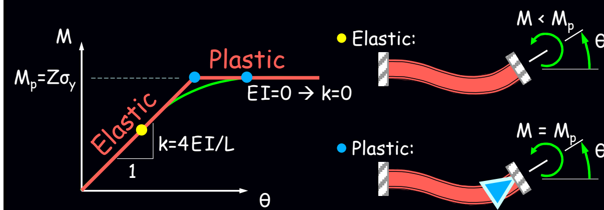








Inelastic Behavior: Flexure

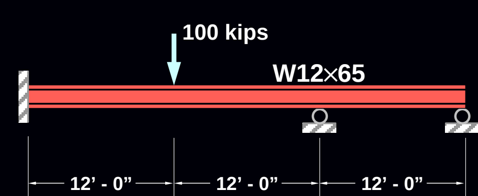


- **Plastic Hinge Model** – Assume section as fully elastic or fully plastic (neglect partial yielding)
- Plastic Hinge at $M = M_p$ or when $M/M_p = 1.0$

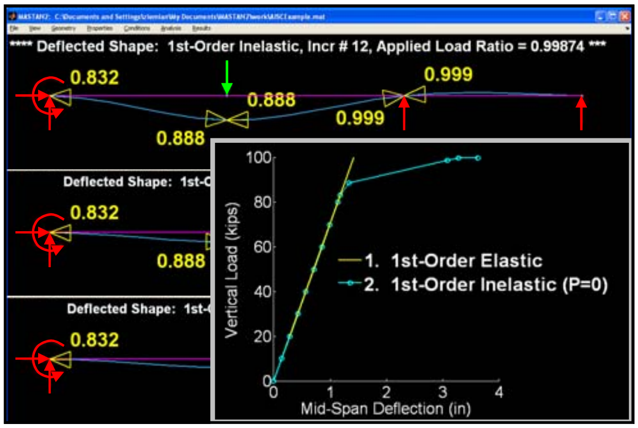
Types of inelastic models

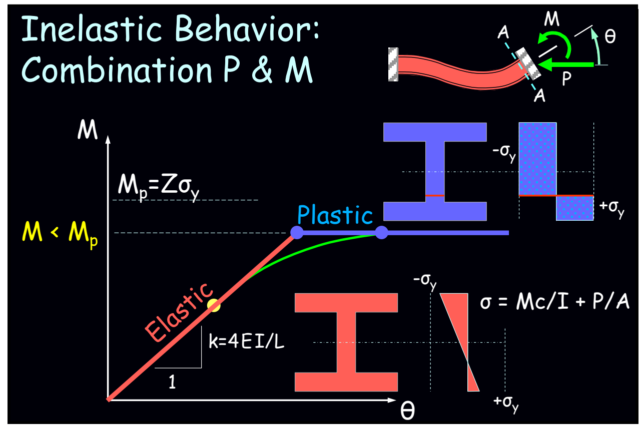
- We will employ a plastic hinge model
  - A.K.A. "Concentrated Plasticity"
  - Section is fully elastic or fully yielded
  - Plastic hinges only at element ends
- Distributed plasticity (still line elements)
  - A.K.A. "Plastic Zone"
  - Captures gradual yielding through depth and along length
  - More accurate, but computationally more $$$
- Finite element with continuum elements ($$$)

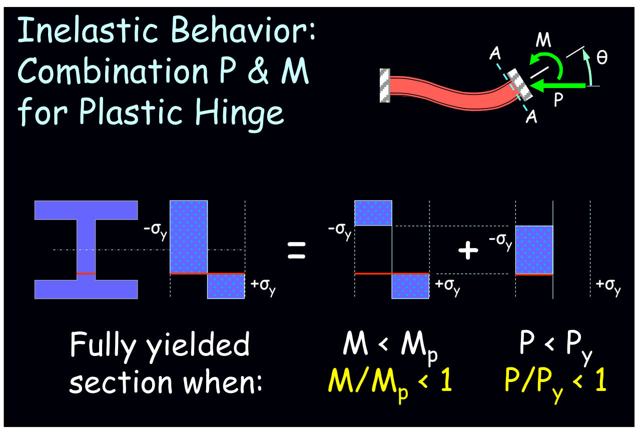
Simple Example:

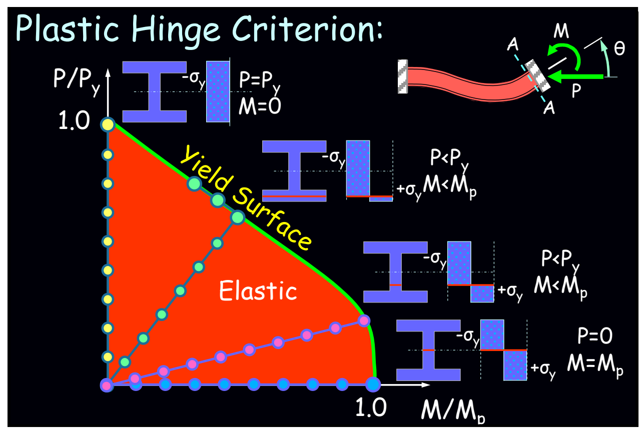


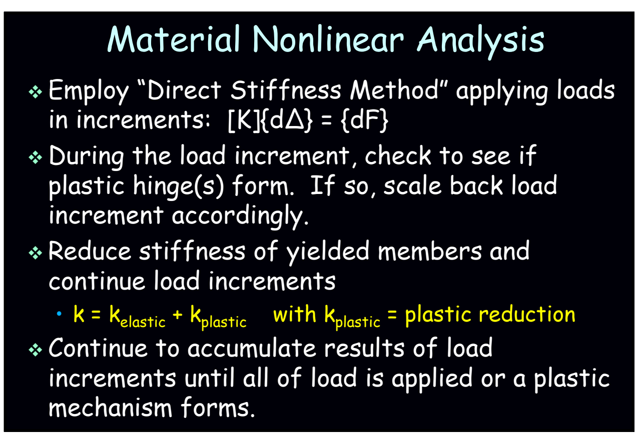
E = 29,000 ksi
σ_y = 50 ksi

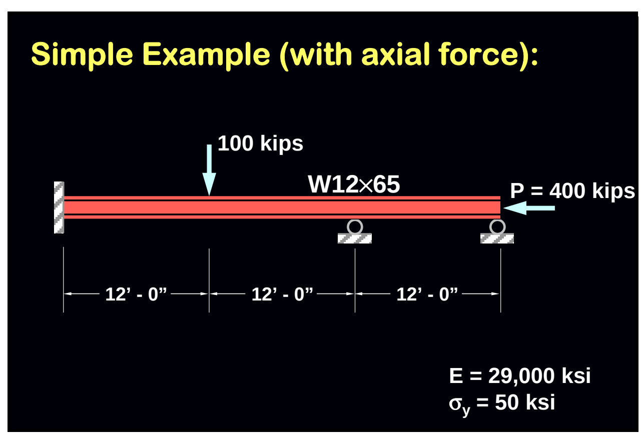

**** Deflected Shape: 1st-Order Inelastic, Incr # 12, Applied Load Ratio = 0.99874 ***
0.832
0.888
0.999
0.999
0.888
Deflected Shape: 1st-O
0.832
0.888
Deflected Shape: 1st-(
0.832
Vertical Load (kips)
Mid-Span Deflection (in)
1. 1st-Order Elastic
2. 1st-Order Inelastic (P=0)

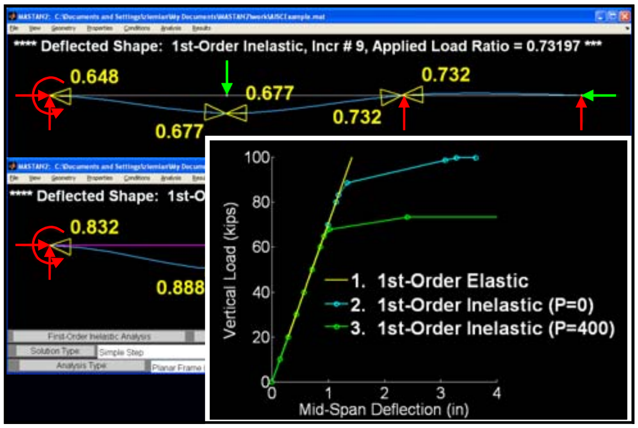

Inelastic Behavior:
Combination P & M
M
θ
P
A
A
$M_p = Z\sigma_y$
M < M_p
Plastic
Elastic
k=4EI/L
1
θ
$-\sigma_y$
$+\sigma_y$
σ = Mc/I + P/A

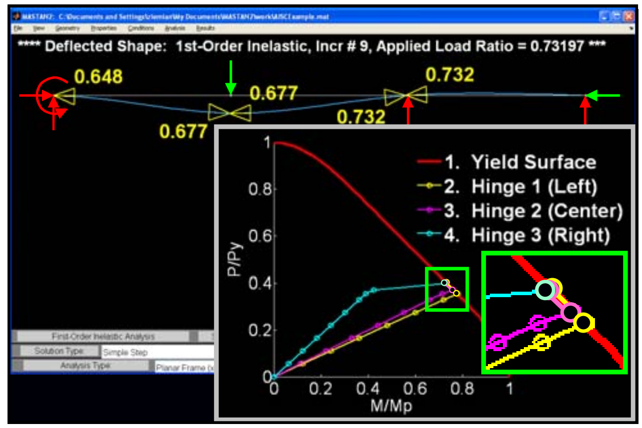

Inelastic Behavior:
Combination P & M
for Plastic Hinge
$-\sigma_y$
$+\sigma_y$
=
+
Fully yielded
section when:
M < M_p
$M/M_p < 1$
P < P_y
$P/P_y < 1$


Plastic Hinge Criterion:
P/P_y
1.0
$P=P_y$
M=0
Yield Surface
$P<P_y$
$M<M_p$
Elastic
$P<P_y$
$M<M_p$
P=0
$M=M_p$
1.0
M/M_p


Material Nonlinear Analysis
Employ "Direct Stiffness Method" applying loads in increments: [K]{dΔ} = {dF}
During the load increment, check to see if plastic hinge(s) form. If so, scale back load increment accordingly.
Reduce stiffness of yielded members and continue load increments
$k = k_{elastic} + k_{plastic}$ with $k_{plastic}$ = plastic reduction
Continue to accumulate results of load increments until all of load is applied or a plastic mechanism forms.


Simple Example (with axial force):
100 kips
W12×65
P = 400 kips
12' - 0"
12' - 0"
12' - 0"
E = 29,000 ksi
σ_y = 50 ksi



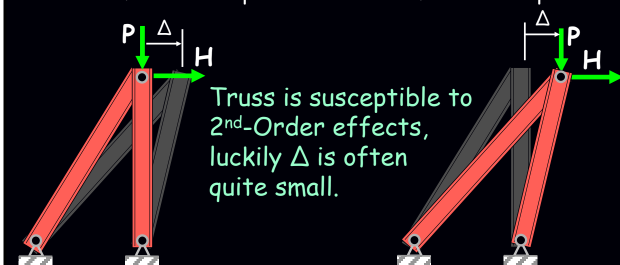


Second-Order Effects

- A.K.A. "Geometric Nonlinear Behavior"
- Equilibrium Equations
  - Reality: Should be formulated on deformed shape
  - Difficulty: Deformed shape (deformations) is a function of the member forces, which are in turn a function of the deformations (Chicken 'n Egg)
  - Remedy: Perform a series of analyses with loads applied in small increments and update geometry after each load increment.

Equilibrium Equations

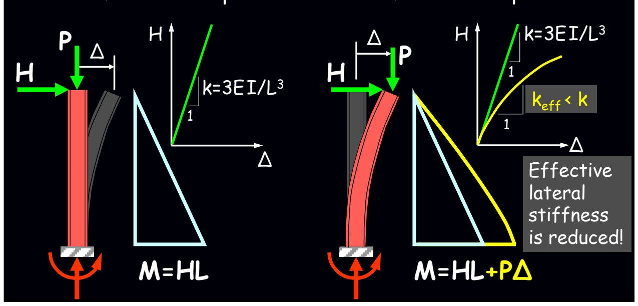
- Formulated on Undeformed Shape
- Formulated on Deformed Shape



Different reactions and member forces.

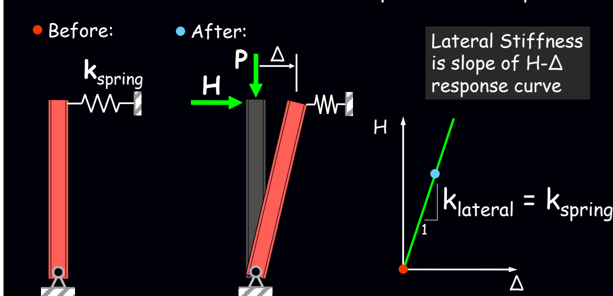
Equilibrium Equations

- Formulated on Undeformed Shape
- Formulated on Deformed Shape



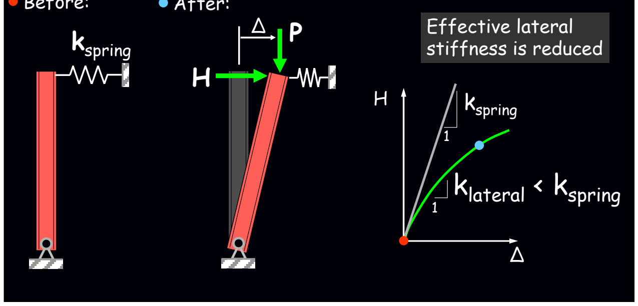
Focus on Lateral Stiffness

- Formulated on Undeformed Shape: Linear Response



Focus on Lateral Stiffness (cont.)

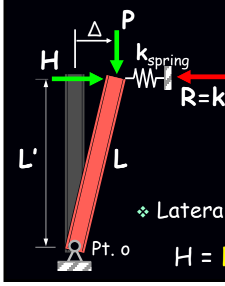
- Formulated on Deformed Shape: Nonlinear Response



Focus on Lateral Stiffness (cont.)

- Equilibrium Formulated on Deformed Shape

Let's start by assuming L' ≈ L,



$\Sigma M_o = 0$ $RL = HL + P\Delta$

$R = H + P\Delta/L$

$k_{spring}\Delta = H + P\Delta/L$

$H = k_{spring}\Delta - P\Delta/L$

$H = (k_{spring} - P/L)\,\Delta$

- Lateral Stiffness (slope of response curve)

$H = k_{lateral}\Delta$ with $k_{lateral} = k_{spring} - P/L$

Some thoughts here…

- This simple analysis becomes less "accurate" as Δ/L becomes large (i.e. Δ/L >> 1/5)
  - Remedy: Perform an incremental analysis and update geometry after each load increment…hence, limit Δ/L in each step to some small amount
  - Keep in mind serviceability limits are often something like Δ/L < 1/400
- Most importantly, $k_{lateral} = k_{spring} - P/L$ takes on the form:

$$k_{2nd\text{-}Order\ El.} = k_{1st\text{-}Order\ El.} + k_g$$

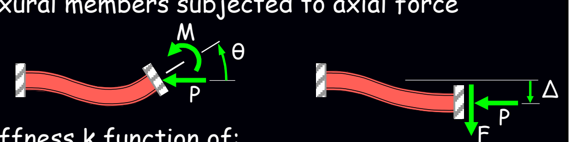
Geometric Stiffness

Geometric Stiffness

- Effective lateral stiffness of a member:
  - decreases as a member is compressed
    - k_g is negative for compressive P
    - backpacker example
  - increases when subjected to tension
    - k_g is positive for tensile P
    - guitar string example
- Employing geometric stiffness approach
  - Other methods exist (i.e. stability functions)

How about real members? (recall…)

- Flexural members subjected to axial force

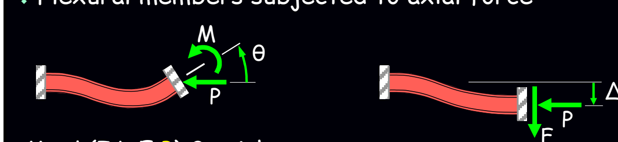


- Stiffness k function of:
  - **Geometry**: Moment of Inertia & Length (I↑,k↑& L↑, k↓)
  - **Material**: Elastic Modulus (E ↑,k↑)
  - **Axial Force**: Compressive (P↑, k↓)

$M = k(I,L,E,P)\,\theta$ $F = k(I,L,E,P)\,\Delta$

Closer look at stiffness terms…

- Flexural members subjected to axial force



$M = k(I,L,E,P)\,\theta$ with
$k = 4EI/L - 2PL/15$

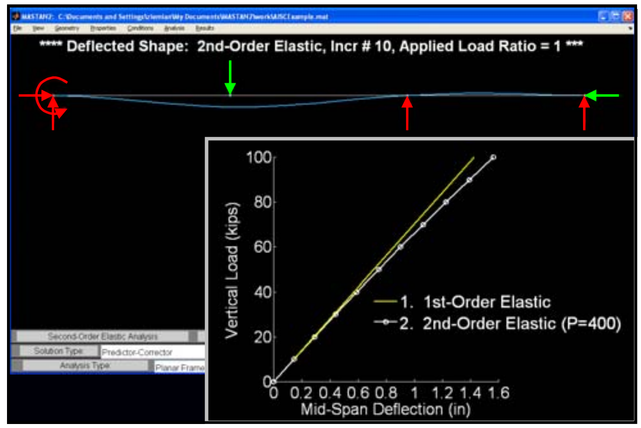
$F = k(I,L,E,P)\Delta$ with
$k = 12EI/L^3 - 6P/5L$

Again, basic form:

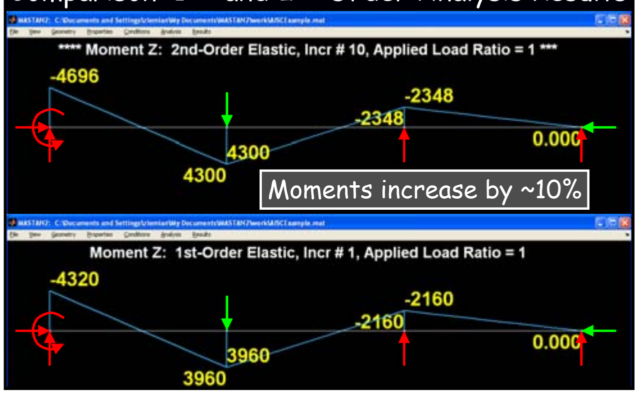
$$k_{2nd\text{-}Order\ El.} = k_{1st\text{-}Order\ El.} + k_g$$

Geometric Nonlinear Analysis

- Employ "Direct Stiffness Method" applying loads in increments: Solve Equil. Eqs. $\{dF\} = [K]\{d\Delta\}$
- At start of increment, modify member stiffness to account for presence of member forces (such as axial force):
  - $k = k_{elastic} + k_g$ with k_g = geometric stiffness
- At end of increment, update model of structural geometry to include displacements
- Continue to accumulate results of load increments ($\Delta_i = \Delta_{i-1} + d\Delta$ and $f_i = f_{i-1} + df$) until all of load is applied or elastic instability is detected.

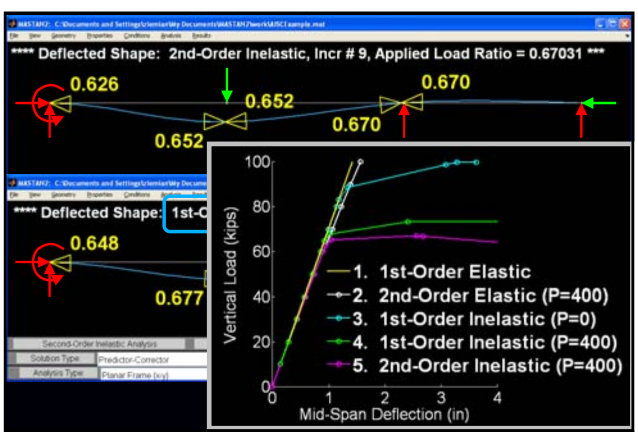


Comparison: 1st- and 2nd-Order Analysis Results



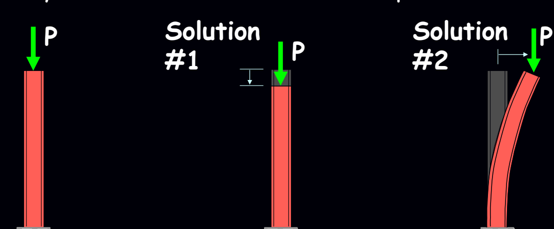
2nd-Order Inelastic Analysis

- Employ "Direct Stiffness Method" applying loads in increments: Solve Equil. Eqs. $\{dF\} = [K]\{d\Delta\}$
- At start of increment, modify member stiffness to account for presence of member forces and any yielding:
  - $k = k_{elastic} + k_{geometric} + k_{plastic}$
- At end of increment, update model of structural geometry to include displacements
- Continue to accumulate results of load increments ($\Delta_i = \Delta_{i-1} + d\Delta$ and $f_i = f_{i-1} + df$) until all of load is applied or inelastic instability is detected.



Critical Load Analysis (Basics)

- Definition: Critical or buckling load is the load at which equilibrium may be satisfied by more than one deformed shape.



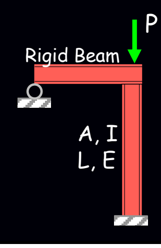
Big Q: How does computer software calculate this?

Critical Load Analysis (Background)

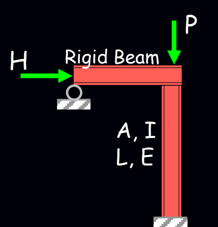
- Elastic stiffness of a member $k = k_{el} + k_g$
  - k_{el} is f(A or I, L, and E)
  - k_g is f(P,L), also note directly proportional to P
- Elastic stiffness of structure $[K] = \Sigma k$
  - $[K] = [K_{el}] + [K_g]$
  - $[K_g]$ directly proportional to applied force
    - i.e. Double applied forces, hence, double internal force distribution and double $[K_g]$
- To the computer, "buckling" will occur when our equilibrium equations $\{F\} = [K]\{\Delta\}$ permit non-unique solutions, e.g. $\det[K] = 0$.

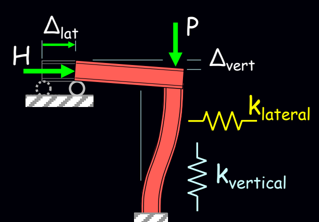
Example

Demonstrate computational method for calculating the elastic critical load (buckling load) for the structural system shown.



Example: Key Stiffness Terms





Vertical Stiffness:

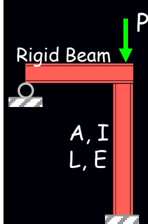
$P = k_{vertical}\,\Delta_{vert}$

Lateral Stiffness:

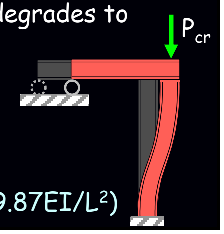
$H = k_{lateral}\,\Delta_{lat}$

$k_{lateral} = 12EI/L^3 - 6P/5L$

Example: Solution



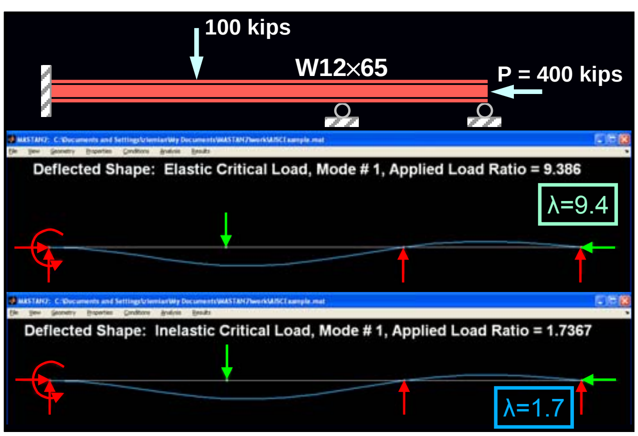
1. Apply reference load, and use 1st-order elastic analysis to obtain internal force distribution.
2. Determine load factor λ at which system stiffness degrades to permit buckling.



$k_{lateral} = 12EI/L^3 - 6\lambda P/5L$

$k_{lateral} = 0$ when $\lambda P = 10EI/L^2$

$P_{cr} = \lambda P = 10EI/L^2$ ($P_{theory} = 9.87EI/L^2$)



Thoughts on Critical Load Analysis

- Computer analysis for a large system:
  - First, apply reference and perform analysis
    - Solve equilibrium eqs. $\{F_{ref}\} = [K]\{\Delta\}$
    - With displacements solve for member forces
  - Second, assemble $[K_{el}]$ and $[K_g]$ based on $\{F_{ref}\}$
  - Finally, determine load factor λ causing instability; computationally this means find load factor λ at which $[K]=[K_{el}]+\lambda[K_g]$ becomes singular
    - Determine λ at which $\det([K_{el}]+\lambda[K_g]) = 0$
    - "Eigenvalue" problem: Eigenvalues = Critical Load Factors, λ's; Eigenvectors = Buckling modes
- Accuracy increases with more elements per compression members (2 often adequate)

Basic Introduction Complete

- Where do I go from here? (Learning to drive)
  - Review the slides (Read the driver's manual)
  - Acquire nonlinear software (Borrow a friend's car)
  - Work lots of examples (Go for a drive, scary at first…)
  - Apply nonlinear analysis in design (NASCAR? not quite)

Acquire nonlinear analysis software
- Commercial programs
- Educational software (i.e. MASTAN2)

MASTAN2:

- Educational software
- GUI ⇔ commercial programs
- Limited # of pre- and post-processing options to reduce learning curve
- Suite of linear and nonlinear 2D and 3D analysis routines
- Available with textbook or online at no cost

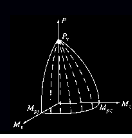
www.mastan2.com or
www.aisc.org [Steel Tools]

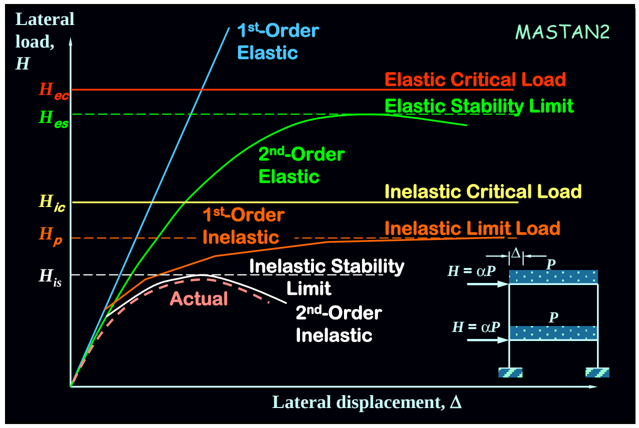
Levels of Analysis: MASTAN2

1st-Order Elastic:	$[K_e]\{\Delta\}=\{F\}$
2nd-Order Elastic:	$[K_e + K_g]\{d\Delta\}=\{dF\}$
1st-Order Inelastic:	$[K_e + K_p]\{d\Delta\}=\{dF\}$
2nd-Order Inelastic:	$[K_e + K_g + K_p]\{d\Delta\}=\{dF\}$
Critical Load:	$[K_e + \lambda K_g]\{d\Delta\}=\{0\}$

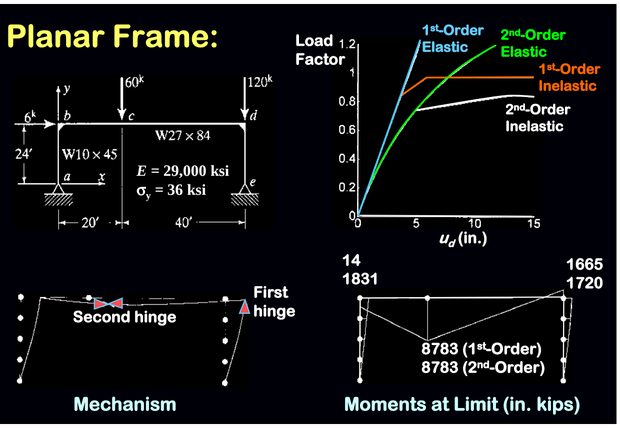
Yield Surface:

Function of P, M_{major}, and M_{minor}





Planar Frame:



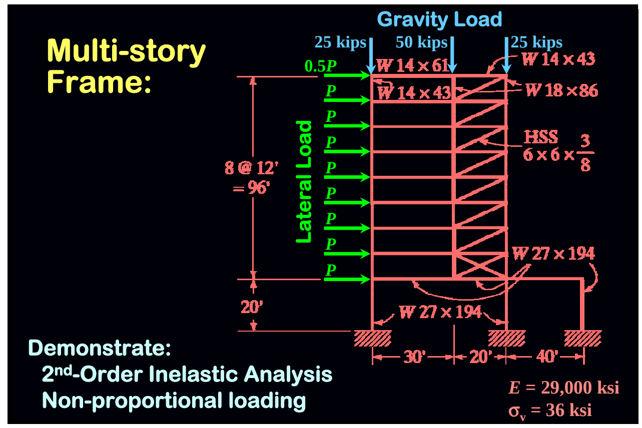
Summary and Conclusions

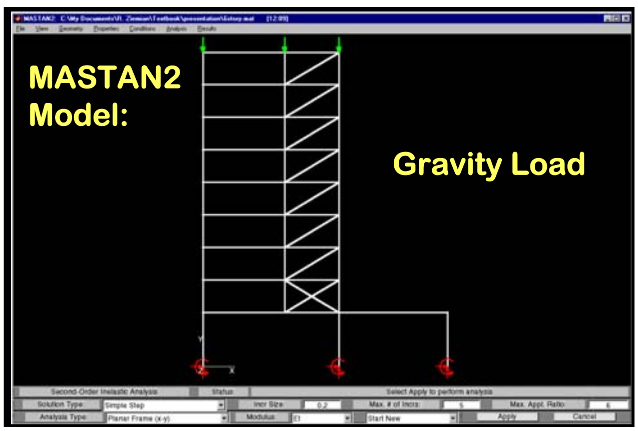
- Provided an introduction to nonlinear analysis
  - Review of direct stiffness method
  - Material nonlinear analysis (Inelastic hinge)
  - Geometric nonlinear analysis (2nd-Order)
  - 2nd-Order inelastic analysis (combine above)
  - Critical load analysis ("eigenvalue analysis")
- Nonlinear…think modifying member stiffness!
- Overview and availability of MASTAN2
- Now, its your turn to take it for a spin…

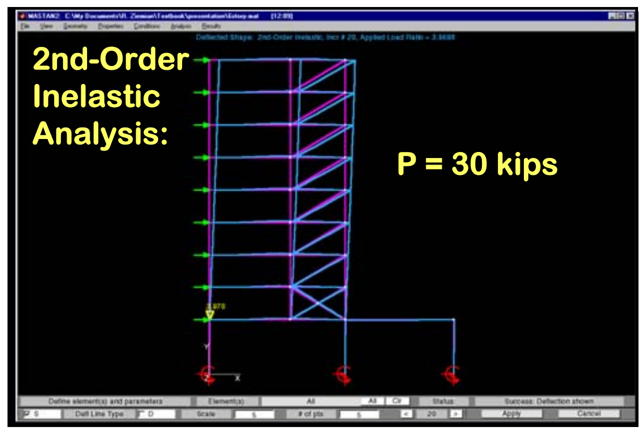
Appendix

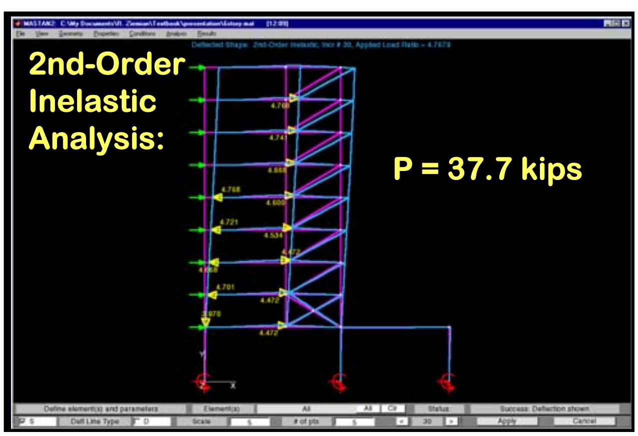
- Several examples to try out
- Solutions by MASTAN2
- Need a reference text with many examples? see Matrix Structural Analysis, 2nd Ed., by McGuire, Gallagher, and Ziemian (Wiley, 2000)
- See tutorial that comes with MASTAN2
- OK, time to jump in and start driving…

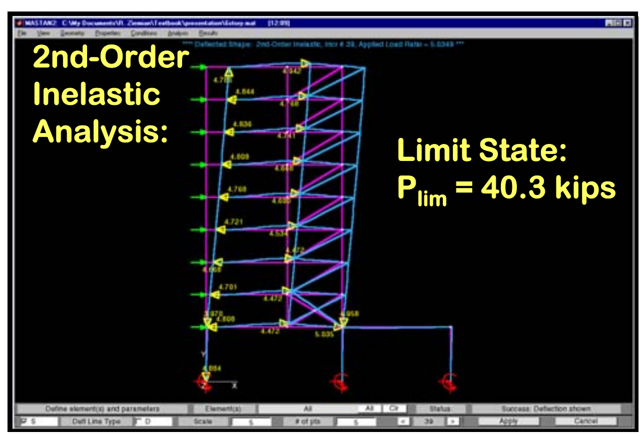
<See Final Exam…>

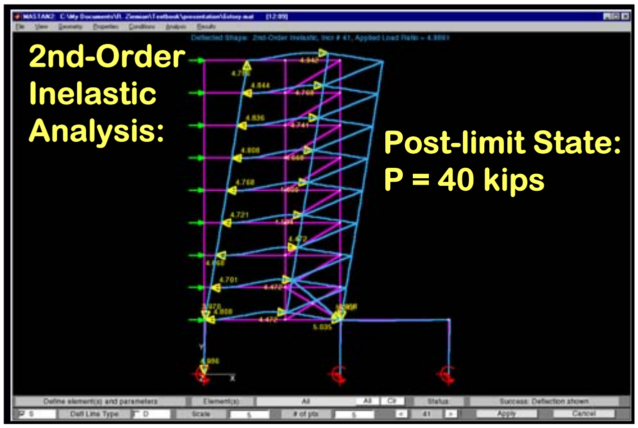


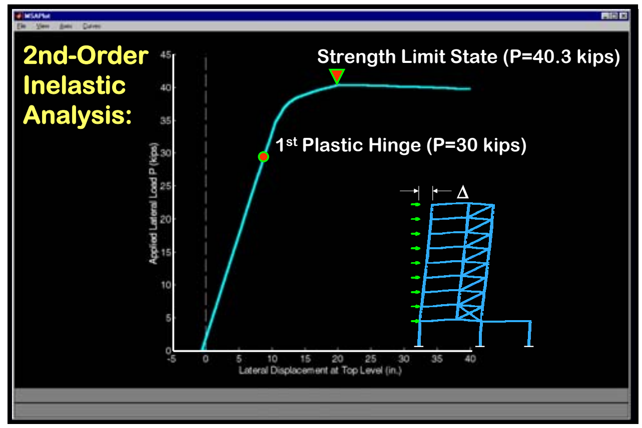


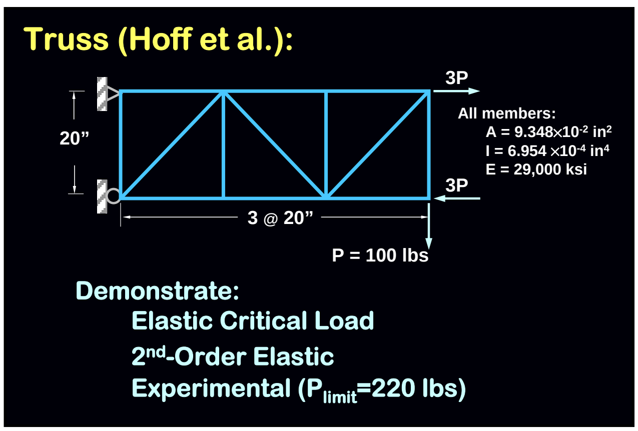


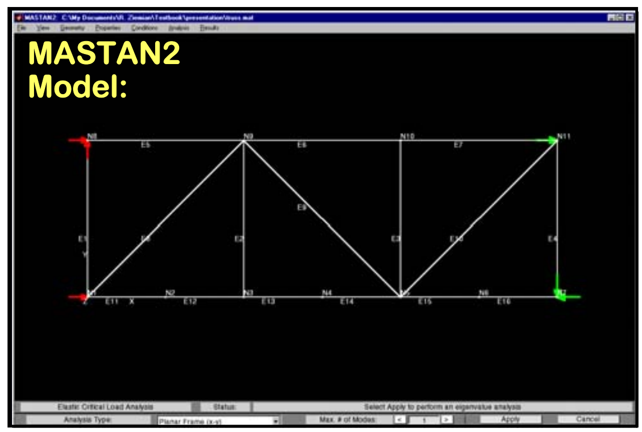


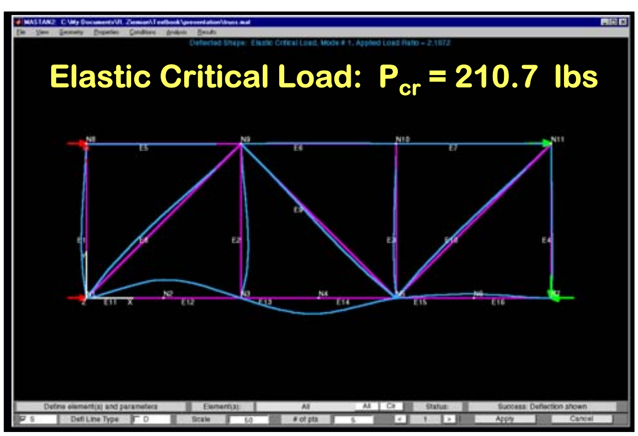


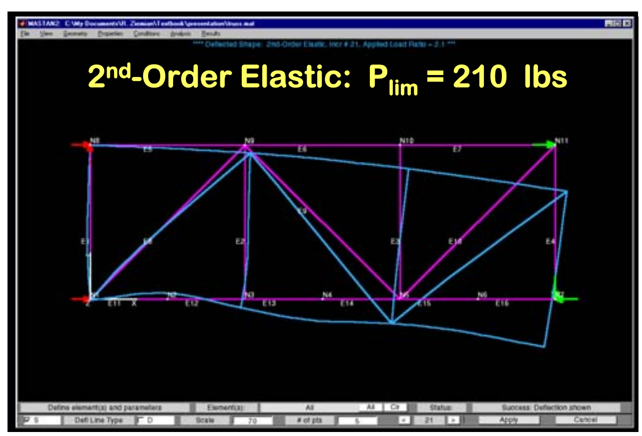

2nd-Order
Inelastic
Analysis:
Post-limit State:
P = 40 kips

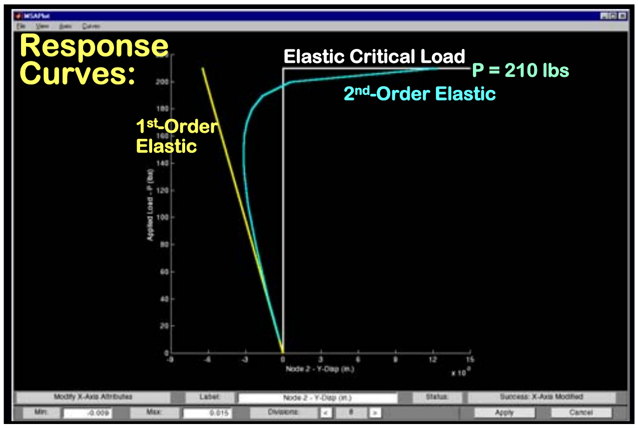

2nd-Order
Inelastic
Analysis:
Strength Limit State (P=40.3 kips)
1st Plastic Hinge (P=30 kips)
Δ
Applied Lateral Load P (kips)
Lateral Displacement at Top Level (in.)

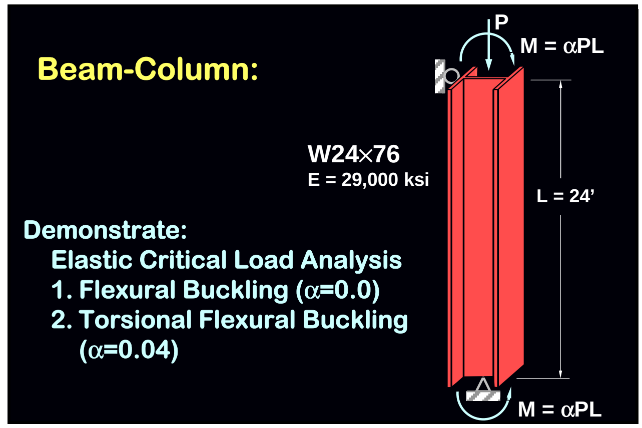

Truss (Hoff et al.):
3P
All members:
A = 9.348×10-2 in2
I = 6.954 ×10-4 in4
E = 29,000 ksi
20”
3P
3 @ 20”
P = 100 lbs
Demonstrate:
Elastic Critical Load
2nd-Order Elastic
Experimental (Plimit=220 lbs)

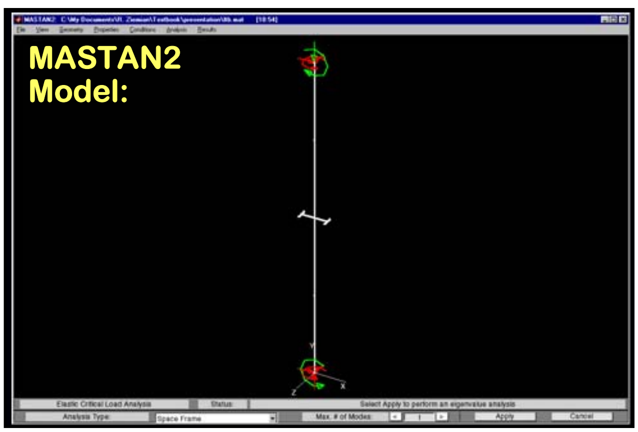

MASTAN2
Model:

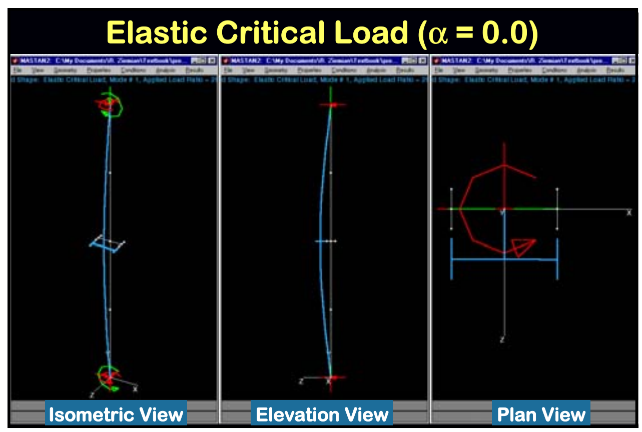

Elastic Critical Load: Pcr = 210.7 lbs

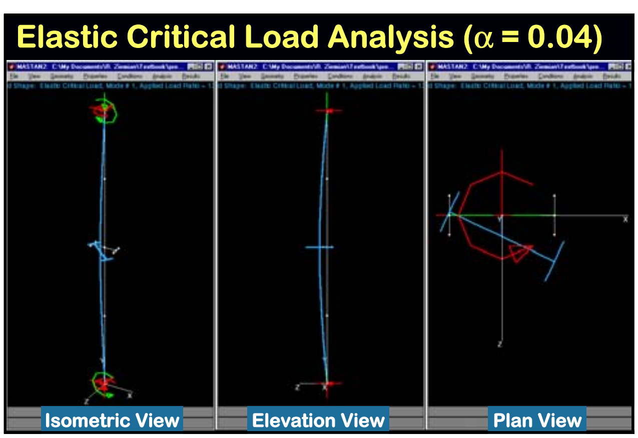

2nd-Order Elastic: Plim = 210 lbs

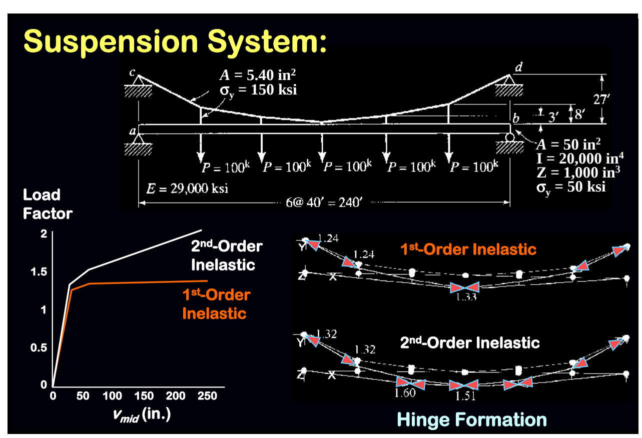

Response Curves:
Elastic Critical Load
P = 210 lbs
2nd-Order Elastic
1st-Order Elastic


Beam-Column:
P
M = αPL
W24×76
E = 29,000 ksi
L = 24'
Demonstrate:
Elastic Critical Load Analysis
1. Flexural Buckling (α=0.0)
2. Torsional Flexural Buckling (α=0.04)
M = αPL


MASTAN2 Model:


Elastic Critical Load (α = 0.0)
Isometric View
Elevation View
Plan View


Elastic Critical Load Analysis (α = 0.04)
Isometric View
Elevation View
Plan View


Suspension System:
A = 5.40 in²
σy = 150 ksi
27'
3'
8'
A = 50 in²
I = 20,000 in⁴
Z = 1,000 in³
σy = 50 ksi
P = 100k
E = 29,000 ksi
6@ 40' = 240'
Load Factor
2nd-Order Inelastic
1st-Order Inelastic
Vmid (in.)
1st-Order Inelastic
2nd-Order Inelastic
Hinge Formation