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In- Plane Stability Considerations of Column Braced Steel Frames

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Abstract
Bracings usually stabilize frame works and prevent- or reduce side sway. They also increase the frame elastic buckling resistance required by design. Yet in modern steel lightweight structures, bracings may be connected to the frame columns in either direction to satisfy architectural and/or structural requirements. Their performance is usually determined by laboratory testing, such as in formworks and infill of structural lightweight panels. This paper presents the exact analytical solution of the elastic buckling of steel portal frames, with bracings connected to columns. The aim is to facilitate the design procedure and to reduce testing costs of individual systems. The governing parameters are: loads, rigidities, stiffness, geometric and boundary conditions. Different model cases are analytically solved and evaluated, critical load values are determined, and accuracy is verified by numerical methods. The interference/interaction between different modes of buckling is determined. The efficiency of column braces is investigated with respect to the different modes of buckling. The best location, geometry and dimensioning of the bracing elements are considered based on the model case parameters. The minimum bracing stiffness is determined to act as stiff bracings. Different cases and their results are presented for direct use of researchers, and simplified methods of solution are given for use of designers including recommendations and precautions.

1. Previous Research

A comprehensive collection of major related research is given by (El-Dib 2009). The bases of the analysis of the elastic buckling of frames are mainly given by (Livesly 1952, Merchant 1955, Goldberg 1968 and Salem 1966,2010). Various stability cases of portal, single bay and multistory frames, un-braced or braced at the corners, are studied in these pioneer researches, giving a wide variety of possible solutions of this problem. Two methods were established and are mainly used: the Direct Method and the Stiffness Method.

2. Method of Solution

The assumptions of the two methods of analysis: the direct and the stiffness methods are:

- The frame material is perfectly elastic.
- Buckling outside the plane of the frame is not considered.

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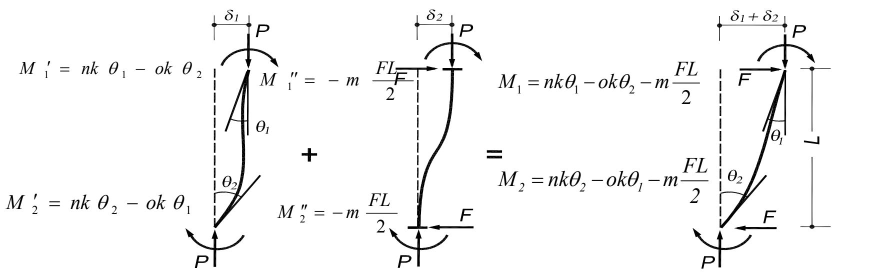
- The deflections of columns and beams are small.
- Columns, beams and bracing members are perfectly straight.
- For all members, the principal axis of bending lies in the plane of the structure, and the shear center coincides with the centroid of the section.
- The effect of the axial deformations produced from second order forces is neglected.
- All loads and displacements are in the plane of frame.
- Bracings are elastic (axial deformations considered); they act in tension and compression.
- The modulus of elasticity E is constant.

2.1 The Direct Method

The following theorem is derived and used (Salem et al, 1966-2010) for combining the different states of buckling deformations: In the presence of axial compression, any general state of sway can be resolved into two states:

1- A state of pure shear sway.
2- A state of no shear sway.

This is valid if the axial force is the same in both cases. The moments and the deflections of the resultant state are equal to the algebraic sum of each of the moments and deflections**:**



$$\frac{\delta_1}{L} = \frac{m}{2}(\theta_1 + \theta_2) \qquad \frac{\delta_2}{L} = \frac{1}{s(1+c)}m\frac{FL}{2k} \qquad \frac{\delta}{L} = \frac{1}{s(1+c)}m\frac{FL}{2k} + \frac{m}{2}(\theta_1 + \theta_2)$$

Figure 1: Resolution of General State of Sway of Intermediate Element in a Column

The non-dimensional functions, used to determine the member end forces and moments (Operators) in Fig.1 are:

$$s = \frac{\mu L\,(\sin\mu L - \mu L\cos\mu L)}{2 - 2\cos\mu L - \mu L\sin\mu L} = \frac{\frac{\mu L}{2}(1 - \mu L\cot\mu L)}{(\tan\frac{\mu L}{2} - \frac{\mu L}{2})}\,;\quad c = \frac{\mu L - \sin\mu L}{\sin\mu L - \mu L\cos\mu L}\,;\quad s'' = s(1 - c^2) = \frac{(\mu L)^2\cdot\sin\mu L}{\sin\mu L - \mu L\cos\mu L}\,;$$

$$m = \frac{2(1 - \cos\mu L)}{\mu L\sin\mu L}\,;\quad n = \mu L\cot\mu L\,;\quad o = \frac{\mu L}{\sin\mu L}\,;\quad \text{and}\quad \mu L = \sqrt{PL/k} = \sqrt{\pi^2\rho}\,, \tag{1}$$

where, the non-dimensional stability functions s, c, s″, m, n and o are functions of the ratio of the axial load to Euler's load: $\rho = P_{CR}/(\pi^2 EI/L^2)$, and are used for hinged base elements as shown in Fig. 2. The stiffness of the element is $k = EI/(Element\ L)$, where I, is the moment of inertia, and the corresponding angle of rotation is considered positive for clockwise rotations. In the state of neutral equilibrium, the external critical loads and the axial forces in the members are in equilibrium. Additional axial and shear forces, and additional moments, occur to restore equilibrium. The relations between the displacements of the joints and the additional forces and moments represent the basic equations for the derivation of the stability condition of the framework.

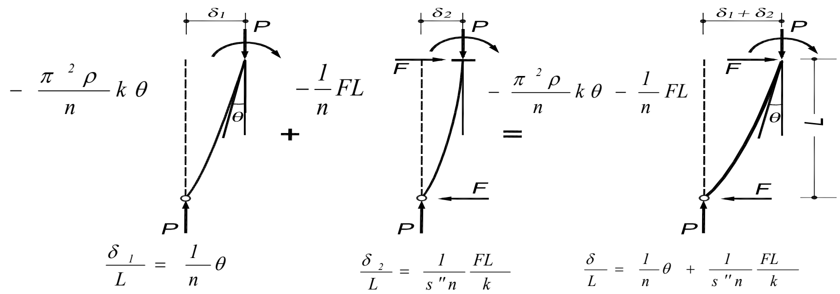


Figure 2: Superposition of States of Sway for Elements with a Hinged End.

The sum of the moments at the frame joints should be zero. This will produce the first set of equations. Also, by equating the relative deflections of the columns at every level, or, at rigidly braced joints, another set of equations can be derived. The stability equations are linear and homogeneous. This means, that no distortions or additional forces occur, and the framework is in a state of stable equilibrium. Values of the unknowns, indicating additional forces and deformations, exist only when the determinant of the system of stability equations vanishes. There is a finite number of solutions of the equation Det = zero, determining an infinite number of different unstable equilibrium configurations. The configuration, which is associated with the smallest value of the load, determines the critical load of the framework. The equation Det.= zero, is solved by trial and error.

2.2 The Stiffness Method

(Hanna 1999) and (Salem et al 2004) applied this method in developing programs, that are used to check individual results obtained from the analytical solution found by the direct method of analysis. Results are found to have a minimum of four digit accuracy.

3. Mode Separation and Case Solution

By separating the general buckling mode into the sway- and the non-sway (symmetric) modes, the solution procedure can be reduced and the accuracy at the modes interface is ensured. The following example demonstrates the solution and the analytical proof of the validity of this separation: By considering the frame in Fig. 3 in the deflected position, five equations of equilibrium in the unknowns: Θ_1 to Θ_4 in addition to "H" could be written as follows:

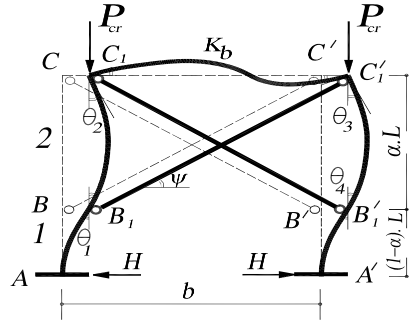


Figure 3: Frame with Corner to Column Stiff Bracings

$\sum M_B$=0; gives : $$\frac{s_2}{\alpha}K\theta_1 - \frac{c_2 s_2}{\alpha}K\theta_2 = n_1\frac{K}{1-\alpha}\theta_1 + \frac{m_1(1-\alpha)}{2}HL; \quad (2)$$

$\Sigma M_C=0$: $$-4K_b\theta_2-2K_b\theta_3=-\frac{s_2}{\alpha}K\theta_2+\frac{c_2s_2}{\alpha}K\theta_1; \tag{3}$$

$\Sigma M_{C'}=0$: $$-4K_b\theta_3-2K_b\theta_2=-\frac{s_2}{\alpha}K\theta_3+\frac{c_2s_2}{\alpha}K\theta_4; \tag{4}$$

$\Sigma M_{B'}=0$: $$\frac{s_2}{\alpha}K\theta_4-\frac{c_2s_2}{\alpha}K\theta_3=n_1\frac{K}{1-\alpha}\theta_4-\frac{m_1(1-\alpha)}{2}HL;\ \text{and} \tag{5}$$

$$\frac{\delta}{L}=-\frac{m_1(1-\alpha)^3}{2s_1(1+c_2)}\frac{HL}{K}+\frac{m_1(1-\alpha)}{2}\theta_1=\frac{m_1(1-\alpha)^3}{2s_1(1+c_2)}\frac{HL}{K}+\frac{m_1(1-\alpha)}{2}\theta_4. \tag{6}$$

After rearranging, the general determinant can be written in the following form:

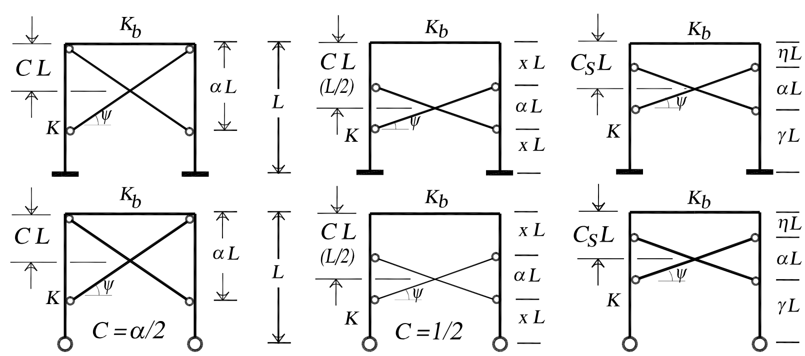
$$\begin{vmatrix} \frac{n_1}{1-\alpha}+\frac{s_2}{\alpha} & \frac{c_2s_2}{\alpha} & zero & zero & zero \\ \frac{c_2s_2}{\alpha} & \frac{s_2}{\alpha}+6\frac{K_b}{K} & zero & zero & zero \\ -\frac{s_1(1+c_1)}{2(1-\alpha)^2} & zero & \mathbf{1} & zero & zero \\ zero & zero & -\frac{m_1(1-\alpha)}{2} & \frac{s_1}{1-\alpha}+\frac{s_2}{\alpha} & \frac{c_2s_2}{\alpha} \\ zero & 2\frac{K_b}{K} & zero & \frac{c_2s_2}{\alpha} & \frac{s_2}{\alpha}+2\frac{K_b}{K} \end{vmatrix} \tag{7}$$

The upper sub-determinant represents the sway, anti-symmetric; mode of buckling, the lower one represents the non-sway (symmetric) mode, and both became independent of each other. By equating each of the sub-determinants to zero, the solution and the interference of the buckling modes could be accurately determined.

4. The Scope

4.1 The General Cases

The three General Bracing Cases in Fig. 4: Top to Column-, Central-, and Intermediate Bracings; are analytically solved, evaluated and presented. They could cover many systems required in the practice. In all studied cases, the selected loads are two concentrated equal loads at the top corners. All frame elements, including bracing elements, are considered elastic.



Bracings: Top to Column Central Intermediate

Figure 4: The selected General System Cases

4.2 The Extreme Cases

The numerical evaluation of general cases may fail at extreme values of parameters. The values of α, η and γ when each is equals null or unity, also when K_b/k in addition to bracing stiffness are extreme, the numerical stability of general case determinants is disturbed. For the purpose of accuracy assurance and additional self checking of all the results, the extreme cases are separately evaluated and represented together with the results of the general cases. Therefore, five determinants, related to extreme cases, are solved for each of the two modes of buckling: the sway and the non-sway modes. Their results are plotted together with the results of the general case to clearly indicate any interference or interaction.

5. Case Study: Typical Buckling Behavior (Fixed Base Frame With Central Column Bracing)

Taking AL^2/I, where I is the column moment of inertia, as a dimensionless parameter in a typical example case solution and result demonstration, explains all the cases presented later (fixed and hinged base frames), and makes it possible to clearly demonstrate how a column braced frame behaves when buckling takes place. When increasing small α-values, according to Fig. 5, bracings with $0 < AL^2/I < 100$ respond slowly. Most efficient are α-values between 0.5 and 0.8. The non-sway (symmetric) mode of buckling interferes over α=0.7, yet bracings can then be more efficient than conventional corner to base braces. It is worth noting that only two general cases (No. 2 and7) are evaluated, all other Extreme Cases are added to get a complete view on the results. In the vicinity of extreme cases the given results have only theoretical value, but are necessary to explain the remarkable "two" results at each of extreme α-magnitudes of zero, or unity. In fact, these "two" ρ-values represent the upper and lower mode bounds of the interaction behavior due to bracing elasticity.

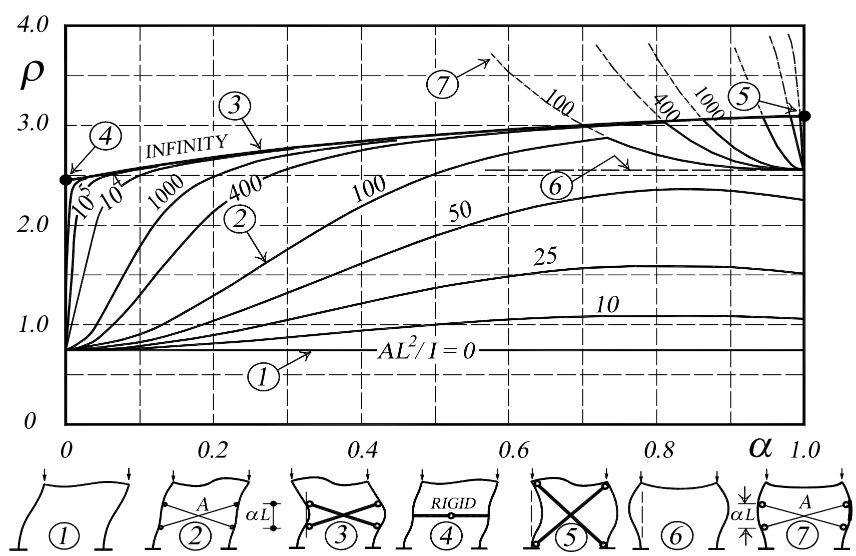


Figure 5: Case Study: Critical ρ-Values for Central Elastic Bracings, $K_b/K=1.0$, $\rho=P_{cr}/(\pi^2 EI/L^2)$, L= Frame Height & Breadth, A: is the Bracing Cross Section Area.

In Fig. 5 the typical general and extreme critical ρ-values are shown and numbered. The seven sketches represent true shapes of the respective buckling first modes, selected out of the ten determinants solved for this case. In the extreme cases No. 3 and 5 no bracing deformations are assumed, the bracings are stiff over all α-values. Case No. 4 is an extreme case of case No. 3, where α=0, and the two bracings coincide together and sway laterally maintaining the same vertical line direction between bracings joints at the column. This case is simply represented by a rigid horizontal member, rigidly connected at both ends to the columns, and located at mid-height of the frame. This is the case of two storey single bay frame, where the weaker story governs the critical buckling load given by the following formula:

$$2n_1 + 6K_b / K = 0. \tag{8}$$

Where, n_1 denotes the upper column segment. In Fig. 5, the point No. 4 matches accurately with the curve No. 3 found from extreme case No.3. The same method is applicable at point No. 5 given by:

$$n + 6K_b / K = 0, \tag{9}$$

where, n represents the whole column, noting that the critical load at point No. 5 is overruled by the line No. 6 (Symmetrical mode, A=0):

$$n + 2K_b / K = 0. \tag{10}$$

6. Frames Braced Top Corner- to Column

As for the top corner- to column braced frame (Fig. 4), the solution determinants of this case are established considering the bracing dimensionless parameter *"B"* (Timoshenco 1936):

$$B = \frac{AL^2}{I} cos^2 \psi \ sin \psi, \tag{11}$$

where, ψ is the bracing inclination with the horizontal (Fig. 4). Using this parameter simplifies the use of the results for frames with different breadth- to height ratios included in the right K_b/K values. Results are verified and demonstrated for some selected parameters (Figs. 6,7). It should be noted that values of K_b/K higher than 4.0 do not add much buckling resistance, and that the most upper limit values of ρ are 4.0 for fixed base- and about 2.0 for hinged base frames.

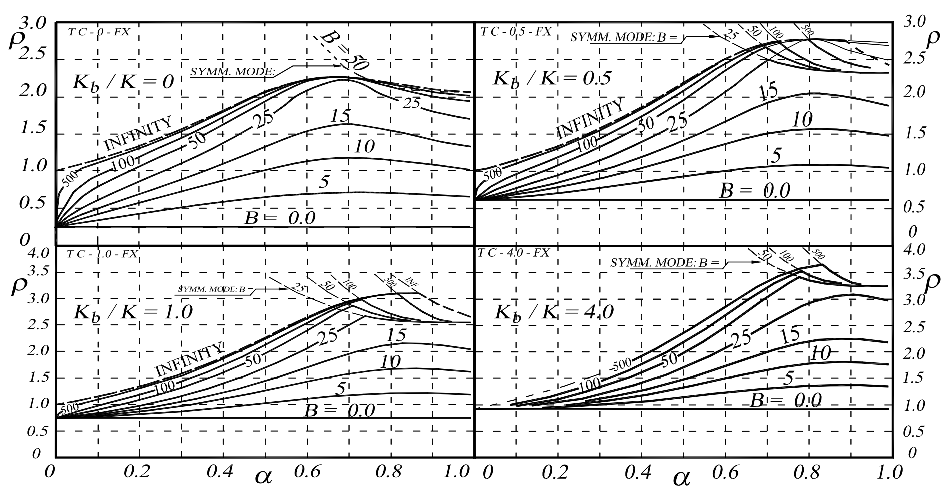


Figure 6: Top Corner- to Column Braced Fixed Base Frame.
Critical ρ-Values, (K_b/K=0, 0.5, 1.0 and 4.0)

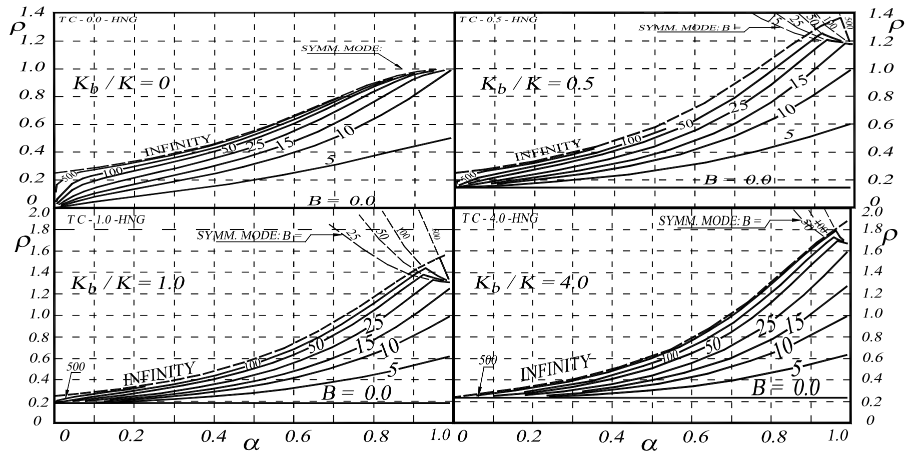


Figure 7: Top Corner- To Column Braced Hinged Base Frame.
Critical ρ-Values. (K_b/K=0, 0.5, 1.0 and 4.0)

7. Frames Braced Centrally to Columns

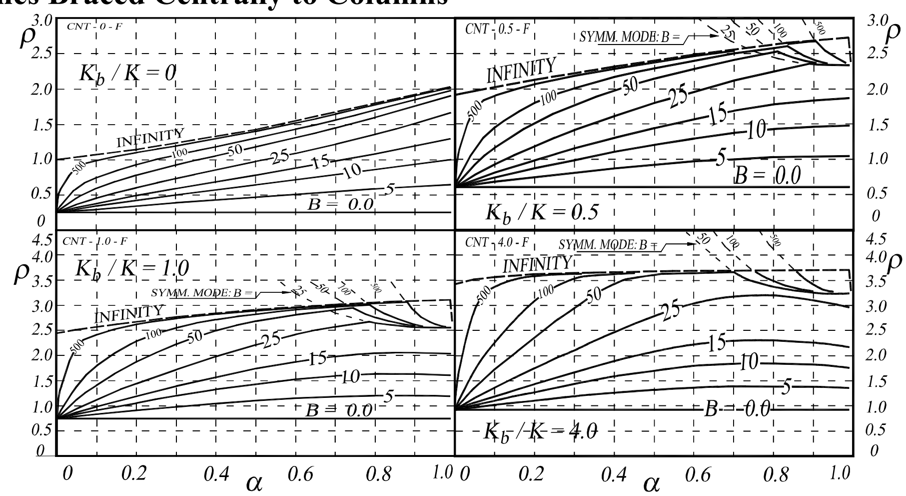


Figure 8: Centrally Column Braced Fixed Base Frame.
Critical ρ-Values. (K_b/K=0, 0.5, 1.0 and 4.0)

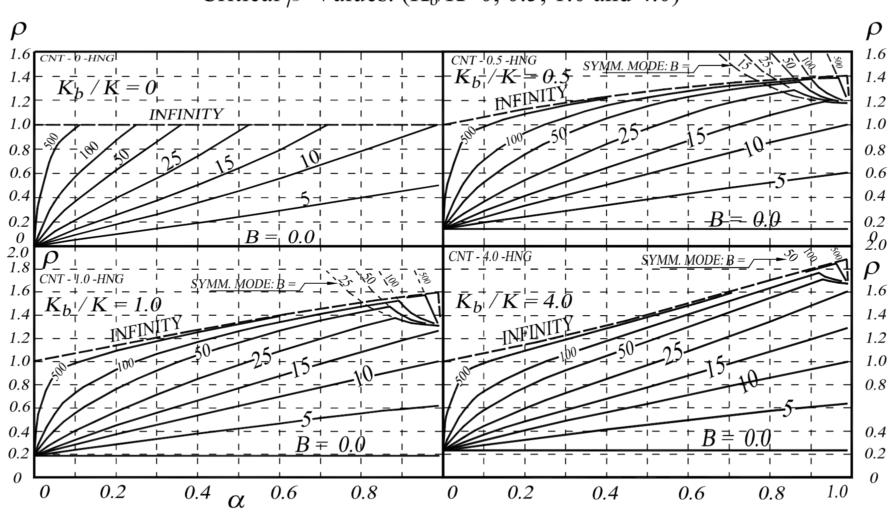


Figure 9: Centrally Column Braced Hinged Base Frame.

The centrally column braced frame is evaluated with the same parameter "*B*" as shown in Fig. 8. This type of bracings can save space and material. In Fig. 9, at $K_b/K=0$, results show a Special Case compared to all other cases. By using the same typical numbering given in Fig. 5, it is noticed that the points similar to No. 4 and 5, in addition to the line No. 3, in Fig. 5, all indicate the same extreme and constant value of ρ_{cr}=1.0 in Fig. 9, which is explained as follows:

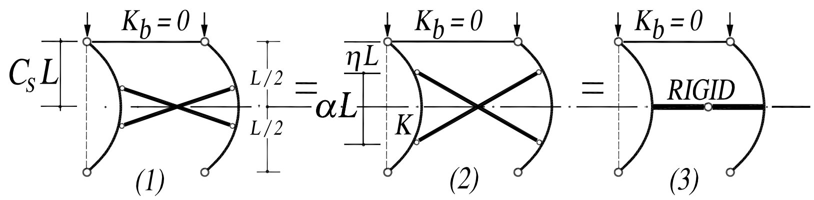


Figure 10: Four Hinged Frame, with Central Rigid Bracings

Fig. 10 represents a hinged frame with K_b=*zero*. In case of sway buckling, stiff central bracings trigger the shown symmetrical shape mode under the critical load. By changing the bracings angle, i.e. α-value, the shape mode doesn't change and remains the same, which is valid for α=1.0, as well as for α=0 as given in Fig. 10-(2),(3). For all these cases, ρ_{cr}=1.0. The position of the bracings centerline in this particular case is called a Special Bracing Location (SBL), at a distance from the top corners:

$$\boldsymbol{C_s} = 1/2 = \eta + \alpha/2. \tag{12}$$

8. Frames with Intermediate Bracings: (Special Bracing Location: SBL)

A Special Bracing Location is defined as the location of stiff bracings centerline, at which the highest possible buckling resistance (ρ_s) could be achieved, and remains unchanged at this location for any bracing angle (αL). This location depends on the frame structural conditions, and the corresponding minimum bracing stiffness depends on the bracing angle (α- value).

8.1 Determination of the Special Bracing Locations (SBL)

The typical extreme results in Fig. 9, which are similar to the points No. 4 and 5, in addition to the line No. 3 in Fig. 5, altogether become special and form a constant (horizontal) upper bound of the critical load dimensionless parameter values of ρ in Fig. 9 at $K_b/K=0$. Therefore, the solution of any of these extreme cases, similar to case No.4, in Fig. 5, leads to the determination of the required SBL of the bracings centerline, and its corresponding extreme ρ_S-values.

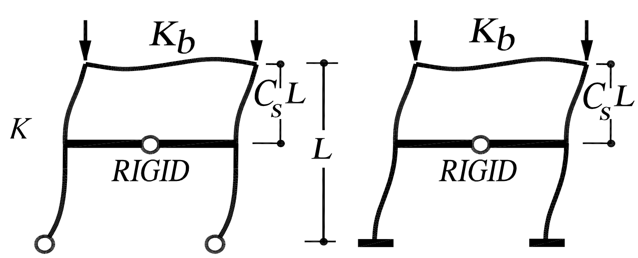


Figure 11: Extreme Case: Intermediate Rigid Beam.

In each of the two storey- single bay frame (Fig. 11), the critical buckling load is governed by the smaller critical storey load. As for either upper story, the equation of the critical buckling load is:

$$n_2 / C_S + 6\, K_b / K = 0 \tag{13}$$

The buckling length of the hinged base lower storey column is simply equal to $\left[2(1-C_S)L\right]$, and for the fixed base one $=\left[(1-C_S)L\right]$. By equating upper- and lower storey critical buckling load values, for each of the given two cases in Fig. 11, an implicit equation in the SBL C_s could be written. Using a two parametric trial and error numerical procedure, the following values of SBL and the corresponding extreme ρ-values are found (ρ_S - values in Table 1).

Table 1: Special Bracing Locations (SBL) C_S and Corresponding ρ_S-Values

Kb/K	C_S	ρ_S	Kb/K	C_S	ρ_S
0.00	0.50000	1.0000	0.00	0.33333	2.2500
0.25	0.55654	1.2712	0.25	0.37213	2.5367
0.50	0.58758	1.4698	0.50	0.39965	2.7745
1.00	0.61796	1.7128	1.00	0.43270	3.1072
2.00	0.64002	1.9292	2.00	0.46121	3.4448
4.00	0.65293	2.0755	4.00	0.47959	3.6924
INF	0.66667	2.2500	INF	0.50000	4.0000
HINGED BASE			FIXED BASE		

With reasonable approximation, the following simplified formulae could be applied to estimate SBL- C_s and the corresponding ρ_S**.** Denoting K_b/K as "κ", we get for fixed base frames:

$$C_S = \frac{2\kappa + 1}{4\kappa + 3},\ and\ \rho_S = \frac{12\kappa + 9}{3\kappa + 4}, \tag{14}$$

and, for hinged base frames:

$$C_S = \frac{4\kappa + 1}{6\kappa + 2},\ and\ \rho_S = \frac{9\kappa + 4}{4\kappa + 4}. \tag{15}$$

8.2 Special Bracings Stiffness

To determine the validity conditions of the special critical buckling loads (ρ_S-values), the total spectrum of the results should be determined for bracings arranged at the SBL "C_S - values" given in Table 1, which are related to K_b/K values. The results plotted in Figs. 12 and 13 demonstrate that, high buckling resistance is achieved not only in case of stiff bracings, but also for any value of bracing elasticity, provided that bracings are placed at the SBL. Given for example that $K_b/K = 4.0$, $\alpha = 0.5$ $(C=0.25)$ and $B=25$, then from Fig.7 $\rho=0.45$. If only C is changed to SBL C_s = 0.65, then ρ_S becomes $=1.75$ (Fig. 13). It is also noticed that, for a given bracing elasticity "B", above a certain value of αL, the extreme ρ_S is reached and maintained, and the bracings act as infinitely stiff. By inspecting these common boundary values versus α and B, it is found that they are almost independent of the stiffness ratio K_b/K. This relationship could be simplified and represented, with reasonable accuracy, as follows:

$$\alpha_s = 3 / \sqrt{B}, \tag{16}$$

where α_s is the minimum bracing angle required for the stiffness B. Eq. 16 is valid for hinged base frames, if 25 <B <500; and for fixed base frames if 35 <B <500. The values of the Special Bracing Angle α_S could also be directly and accurately taken from Figs. 12-13.

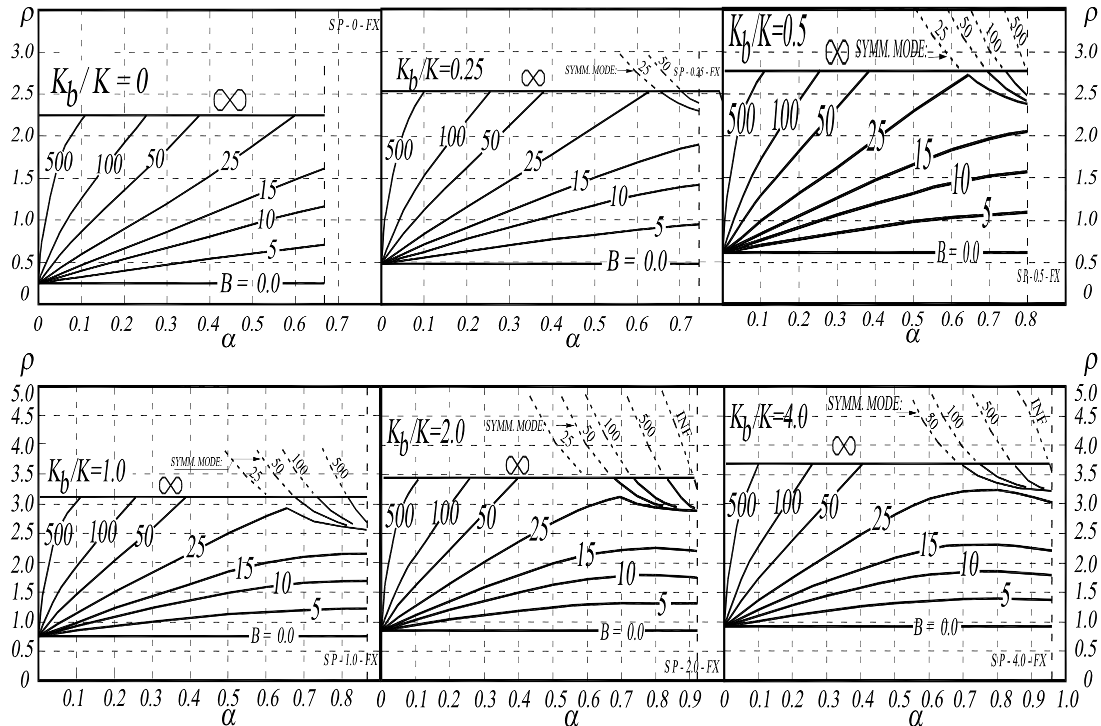


Figure 12: Special Critical ρ_S- and ρ-Values at SBL C_S (Table 1, Fixed Base)
Related to K_b/K, B and α.

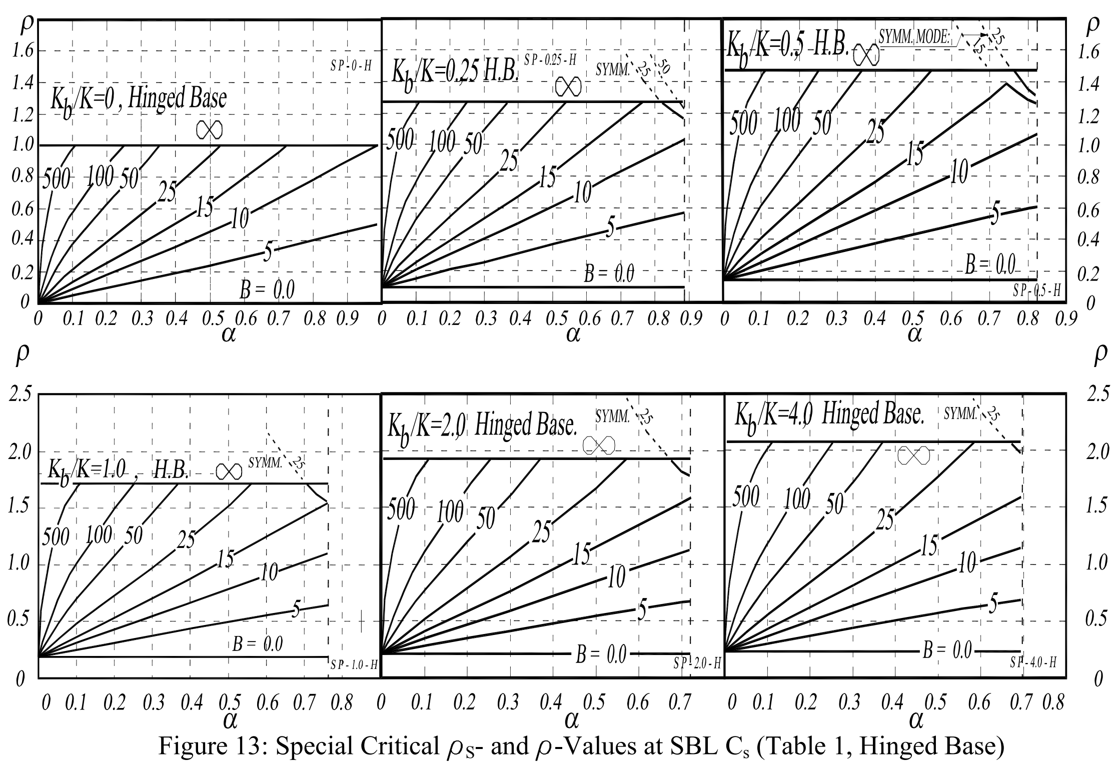


Figure 13: Special Critical ρ_S- and ρ-Values at SBL C_s (Table 1, Hinged Base)
Related to K_b/K, B and α.

9. Solved Example: Sway Buckling

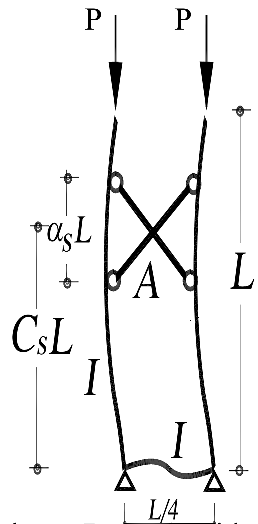


Figure 14: Column Braced Bridge Pylon

The hinged base frame with intermediate bracing in Fig. 4, when rotated by 180^0, is equivalent to the case given in Fig. 14. The loads in this case are assumed vertical and are maintaining their vertical direction of action in the deformed position. Given in this case $K_b/K=4.0$; the SBL, from Table 1, is at $C_S = 0.65293$. The critical sway buckling load ratio, which is related to this location, is: $\rho_S = P_{cr}/(\pi^2 EI/L^2) = 2.075$, can only be secured, if, and only if the relationship between α_S and B is maintained, either from Eq. 16, or directly from Fig. 13 (for $K_b/K=4$). Select $B=25$, the corresponding minimum α from Eq.16 is $\alpha_S = 0.6$. We could also select $B=500$ and its $\alpha_S = 0.13$, which is uneconomic and not recommended. Using $B=25$, determine the required minimum (effective) bracing area "A" from Eq. 11.

Another solution is to maintain the Special Location at $C_S=0.65293$ and to select $B=10$ (instead of 25), keeping $\alpha_S=0.6$, then ρ becomes **= *1.0*** (Fig. 13)**.** Such solution is economical, provided that it fulfills the structural requirements.

A comparison between Bracing Locations of all three General Cases (Fig. 4) is given, for hinged base frames, in Table 2, which clearly demonstrates the efficiency of placing the bracings centerline at the Special Bracings Locations (SBL).

Table 2: Critical Values for Different Bracing Types (Hinged Base, $K_b/K=4$)

Bracing type	C , C_S	α	B	ρ , ρ_S
Top Corner- to Column (Fig. 7)	0.3	0.6	25	0.557
Central (Fig. 9)	0.5	0.6	25	1.092
SBL	C_S=0.6529	0.6*	25*	ρ_S =2.075
SBL	C_S=0.6529	0.5*	36*	ρ_S =2.075
SBL	C_S=0.6529	0.4*	56.3*	ρ_S =2.075
SBL	C_S=0.6529	0.3*	100*	ρ_S =2.075

* According to Eq. 16.

To make full use of the SBL just use either Eq. 14 or 15, in addition to Eq. 16. Alternatively, use Figs. 12 or 13. Then place, construct and execute bracing centerline only at SBL.

10. Conclusions

Column Bracings can take several forms. Presented cases showed that economical solutions could be found that remarkably increase the buckling resistance of frames with column braces. The bracings can be connected to top corners or arranged centrally. They could also be placed at Special Bracing Locations (SBL) C_S to obtain the best possible increase in buckling resistance of the available system. Accurate solutions may directly be obtained from the plotted results, or calculated from given simplified equations.

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