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Analysis of locally/distortionally buckled beams

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Abstract
The paper describes the derivation of a seven degrees of freedom beam finite element which enables the effects of local/distortional buckling deformations to be accounted for. The development of local/distortional buckling reduces the rigidity of the section against axial straining, minor and major axis flexure, as well as twisting. The reduction in rigidity can be determined by increasing the level of axial strain, minor axis curvature, major axis curvature or twist, and at each level of deformation subjecting a single or a few local/distortional buckles to small changes in axial compression, minor and major axis bending, and torsion. This analysis is performed prior to the frame analysis and produces arrays of tangent rigidities ($(EA)_t$, $(EI_z)_t$, $(EI_y)_t$, $(EI_w)_t$) and other tangential stiffness terms for increasing values of generalised strains ($\varepsilon_x, \kappa_z, \kappa_y, \kappa_x$). Incorporating the reduced rigidities in the beam element formulation requires changes to the tangential stiffness matrix.

The seven degree of freedom element is developed in the framework of the OpenSees software. The paper sets out the tangential stiffness matrix for a locally/distortionally buckled element and shows that close agreement can be obtained between the beam analysis which incorporates reduced rigidities and analysis using full shell finite element discretisation. The purpose of the developed element is to make beam-element analyses readily available for analysing the structural response of locally/distortionally buckled frames. This is particularly relevant for determining the additional second order moments resulting from the increased sway induced by the reduction in flexural and warping rigidity. This effect is becoming increasingly important to quantify as cold-formed steel sections are being produced in increasingly thinner gauges and subject to local/distortional buckling in the ultimate limit state. Yet, the associated additional second order moments are presently not considered in the AISI S100-2007 Specification for the design of cold-formed steel structural members.

1. Introduction

Beam-element-based analyses of structures are regarded as general and practical tools for actual industrial use, and have been proven capable of accurately predicting ultimate loads and tracking load-displacement curves for compact sections (Ziemian, R. D. 1990). However, this is not the

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case for structures with non-compact and slender cross-sections, because traditional beam-column element theories assume that the cross-section remains undistorted throughout the analysis, and thus are not capable of considering deformations of the cross-section.

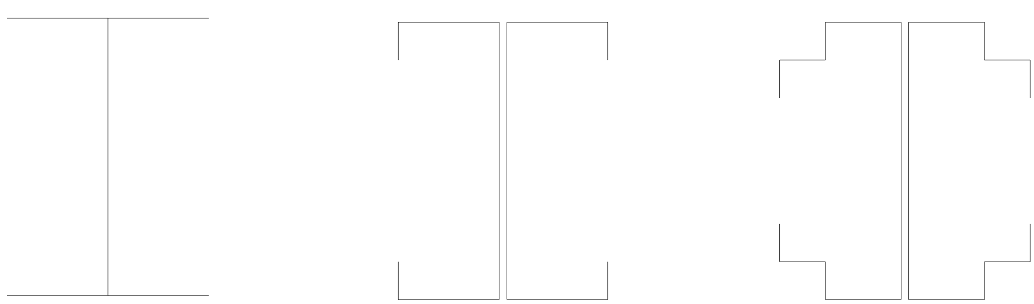
The primary effect of local/distortional buckling can be perceived as merely the reduction of the member stiffness against overall compression, bending and torsion. Consequently, the overall behaviour of the structure can be achieved by using the stiffness of the locally/distortionally buckled cross-section rather than the stiffness of the undistorted cross-section in beam element analyses. This method has been successfully used for bifurcation analyses of locally buckled members (Rasmussen, K. J. R. and Hasham, A. S. 1997; Young, B. and Rasmussen, K. J. R. 1997). This paper presents a method to include local/distortional buckling effects in geometrically nonlinear beam-element-based analyses where local/distortional buckling deformations are taken account of by simply reducing the rigidities of the section. The reduction of the tangent rigidities are determined by means of *a priori* finite element analyses of short lengths of members.

The Open System for Earthquake Engineering System (OpenSees) (Mckenna, F. T. 1997) was chosen as the research tool to perform beam element analysis. As a first step, the original OpenSees source codes were modified to include warping effects of open compact sections. Details can be found in (Zhang, X., Rasmussen, K. J. R. and Zhang, H. 2012). In this paper, the analysis is developed further to allow for local/distortional deformations.

The accuracy of the presented beam element analysis program accounting for local/distortional buckling is checked against benchmark problems analysed separately using full shell finite element discretisation in (ABAQUS 2009). Good agreement is achieved for both single members and frames.

2. Application of cross-sections

The presented analysis method is applicable to doubly symmetric open sections that may develop local/distortional buckling. Examples of application are I-sections (Figure 1(a)), back-to-back lipped channel sections (Figure 1(b)) and other non-regular sections (Figure 1(c)).



(a) (b) (c)

Figure 1: Examples of applicable cross-sections

3. Local and global systems

The present theory utilizes the co-rotational Lagrangian (CL) approach to formulate non-linear beam-column elements. In this approach, the deformational response is captured at the level of the local reference frame, whereas the geometric non-linearity induced by large displacements is considered in the transformation matrices relating local and global quantities.

The vector of global degrees of freedom $\mathbf{N}$ and the vector of local degrees of freedom $\mathbf{n}$ are defined as follows,

$$\mathbf{N}=\{u_1,v_1,w_1,\omega_1,\omega_2,\omega_3,\theta_{b1}^{'},u_2,v_2,w_2,\omega_4,\omega_5,\omega_6,\theta_{b2}^{'}\} \tag{1}$$

$$\mathbf{n}=\{\theta_{l1},\theta_{l2},\theta_{l3},\theta_{b1}^{'},\theta_{l4},\theta_{l5},\theta_{l6},\theta_{b2}^{'},e\} \tag{2}$$

where the components of the vectors are shown in Figure 2. Note that θ_{b1}' and θ_{b2}' are the warping degrees of freedom.

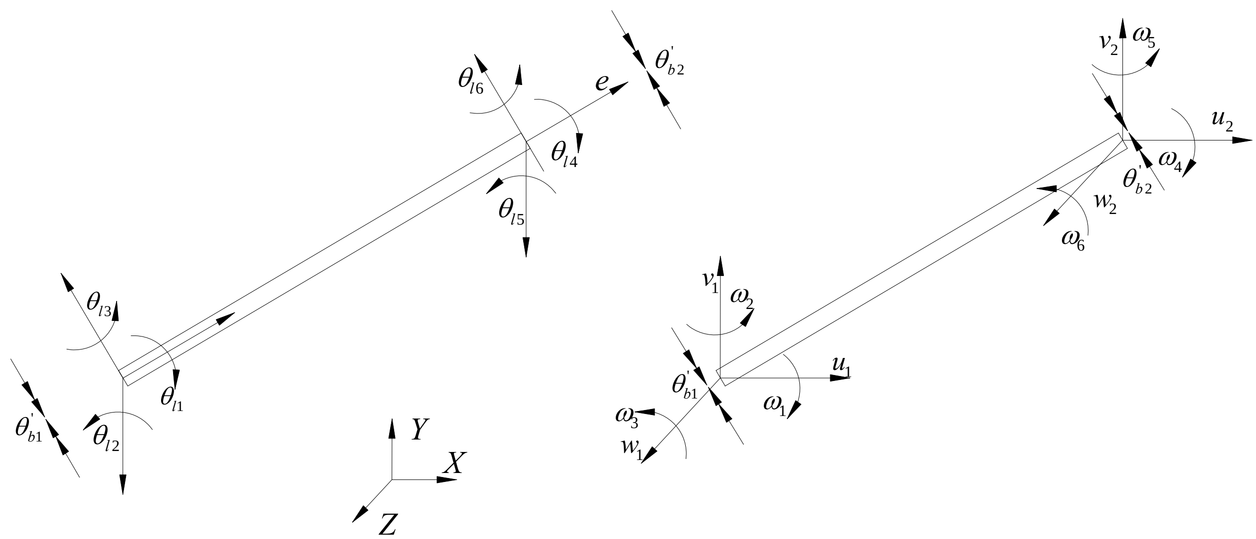


Figure 2: Element local and global degrees of freedom

4. Formulation of stiffness matrix with consideration of local/distortional buckling

In the modified OpenSees codes, the axial strain (ε) is expressed as (Alemdar, B. N. 2001),

$$\varepsilon = u' + \frac{1}{2}(v')^2 + \frac{1}{2}(w')^2 - \varpi\varphi'' - yv'' - zw'' + \frac{1}{2}(y^2+z^2)(\varphi')^2 - y\varphi w'' + z\varphi v'' \tag{3}$$

where u,v,w are the local displacements of the shear centre of the cross-section, φ is the twist rotation about the shear centre, (the centroid and the shear centre coincide for doubly-symmetric section), y,z are the local coordinates of an arbitrary point of the cross-section in the principal axis system, and $\varpi = \varpi(y,z)$ is the principal sectorial coordinate of the arbitrary point. The local coordinate axes (y,z) and the local displacements of shear centre (u,v,w) are shown in Figure 3.

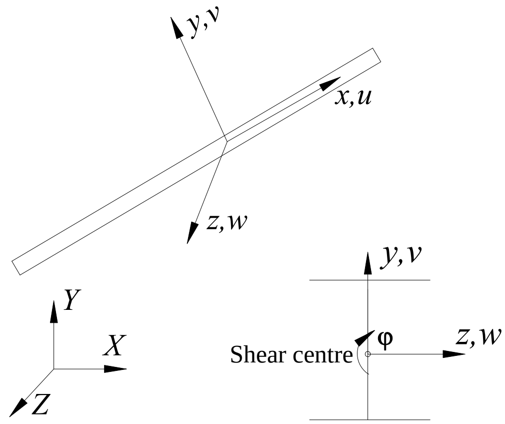


Figure 3: Coordinate axes and shear centre displacements

The variation of ε is expressed as,

$$\delta\varepsilon = \delta u' + v'\delta v' + w'\delta w' - y(\delta v'' + w''\delta\varphi + \varphi\delta w'') + z(-\delta w'' + v''\delta\varphi + \varphi\delta v'') + (y^2 + z^2)\varphi'\delta\varphi' - \varpi\delta\varphi'' \quad (4)$$

In the present formulation for thin-walled sections, shear strains due to bending and warping torsion are neglected, and the shear strain due to uniform torsion is assumed to vary linearly through the thickness of component plates with zero mid-plane value (Rasmussen, K. J. R. 1997). The shear strain due to uniform torsion is given by (Pi, Y. L., Trahair, N. S. and Rajasekaran, S. 1992)

$$\gamma = -2n\kappa_z \quad (5)$$

where n is a coordinate perpendicular to the tangent of the mid-surface at an arbitrary point and κ_z is the twist,

$$\kappa_z = \varphi' + \frac{1}{2}\left(v''w' - v'w''\right) \quad (6)$$

By assuming the twist due to bending can be ignored, the shear strain due to uniform torsion can be approximated by

$$\gamma = -2n\varphi' \quad (7)$$

and its variation is

$$\delta\gamma = -2n\delta\varphi' \quad (8)$$

The normal and shear strains are assembled in the strain vector,

$$\boldsymbol{\varepsilon} = [\varepsilon, \gamma]^T \quad (9)$$

The variations of the strain components may be expressed in matrix form,

$$\delta\boldsymbol{\varepsilon} = \begin{bmatrix} \delta\varepsilon \\ \delta\gamma \end{bmatrix} = \mathbf{Y}\,\delta\boldsymbol{\Gamma} \tag{10}$$

where **Y** and δ**Γ** are functions of the coordinates (y,z,ϖ,n) and generalised displacements (u,v,w,φ), respectively,

$$\mathbf{Y} = \begin{bmatrix} 1 & y & z & y^2+z^2 & \overline{\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2n \end{bmatrix} \tag{11}$$

$$\delta\boldsymbol{\Gamma} = \begin{bmatrix} \delta\varepsilon_x \\ -\delta\kappa_z \\ \delta\kappa_y \\ \delta\kappa_W \\ \delta\kappa_x \\ -\delta\varphi' \end{bmatrix} = \begin{bmatrix} \delta u' \\ -\delta v'' \\ -\delta w'' \\ 0 \\ -\delta\varphi'' \\ -\delta\varphi' \end{bmatrix} + \begin{bmatrix} v'\,\delta v' + w'\,\delta w' \\ -\varphi\,\delta w'' - w''\,\delta\varphi \\ \varphi\,\delta v'' + v''\,\delta\varphi \\ \varphi'\,\delta\varphi' \\ 0 \\ 0 \end{bmatrix} \tag{12}$$

The variation of the virtual generalised strain vector (δ**Γ**) is expressed as,

$$\delta\boldsymbol{\Gamma} = \mathbf{N}_{\delta d1}\,\delta\mathbf{v} \tag{13}$$

where

$$\mathbf{N}_{\delta d1} = \begin{bmatrix} 1 & v' & w' & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & -\varphi & -w'' & 0 & 0 \\ 0 & 0 & 0 & \varphi & -1 & v'' & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \varphi' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} \tag{14}$$

$$\delta\mathbf{v} = \begin{bmatrix} \delta u' & \delta v' & \delta w' & \delta v'' & \delta w'' & \delta\varphi & \delta\varphi' & \delta\varphi'' \end{bmatrix}^T \tag{15}$$

In developing an equation for δ**v**, shape functions are introduced to express the axial displacement (u), the transverse displacements (v,w) and the twist rotation (φ) as continuous functions of the longitudinal coordinate (x). A linear function is chosen for the axial displacement, while cubic Hermitian functions are chosen for the transverse displacements and twist rotation,

$$
\begin{aligned}
u &= \mathbf{N}_u^{\mathbf{T}}\mathbf{n} \\
v &= \mathbf{N}_v^{\mathbf{T}}\mathbf{n} \\
w &= \mathbf{N}_w^{\mathbf{T}}\mathbf{n} \\
\varphi &= \mathbf{N}_\phi^{\mathbf{T}}\mathbf{n}
\end{aligned}
\tag{16}
$$

where **n** is the element nodal displacement vector in the local system, given by Eqn. (2).

The shape functions can be expressed in the following form,

$$
\begin{aligned}
\mathbf{N}_u^{\mathbf{T}} &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad N_{u1}] \\
\mathbf{N}_v^{\mathbf{T}} &= [0 \quad N_{v1} \quad 0 \quad 0 \quad 0 \quad N_{v2} \quad 0 \quad 0 \quad 0] \\
\mathbf{N}_w^{\mathbf{T}} &= [0 \quad 0 \quad N_{w1} \quad 0 \quad 0 \quad 0 \quad N_{w2} \quad 0 \quad 0] \\
\mathbf{N}_\varphi^{\mathbf{T}} &= [N_{\varphi1} \quad 0 \quad 0 \quad N_{\varphi2} \quad N_{\varphi3} \quad 0 \quad 0 \quad N_{\varphi4} \quad 0]
\end{aligned}
\tag{17}
$$

With the aid of Eqn. (16), $\delta\mathbf{v}$ can be expressed using the element nodal displacement vector **n**,

$$
\delta\mathbf{v} = \mathbf{N}_{\delta d2}\delta\mathbf{n} \tag{18}
$$

where

$$
\mathbf{N}_{\delta d2} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_{u1}^{'} \\
0 & N_{v1}^{'} & 0 & 0 & 0 & N_{v2}^{'} & 0 & 0 & 0 \\
0 & 0 & N_{w1}^{'} & 0 & 0 & 0 & N_{w2}^{'} & 0 & 0 \\
0 & N_{v1}^{''} & 0 & 0 & 0 & N_{v2}^{''} & 0 & 0 & 0 \\
0 & 0 & N_{w1}^{''} & 0 & 0 & 0 & N_{w2}^{''} & 0 & 0 \\
N_{\varphi1} & 0 & 0 & N_{\varphi2} & N_{\varphi3} & 0 & 0 & N_{\varphi4} & 0 \\
N_{\varphi1}^{'} & 0 & 0 & N_{\varphi2}^{'} & N_{\varphi3}^{'} & 0 & 0 & N_{\varphi4}^{'} & 0 \\
N_{\varphi1}^{''} & 0 & 0 & N_{\varphi2}^{''} & N_{\varphi3}^{''} & 0 & 0 & N_{\varphi4}^{''} & 0
\end{bmatrix} \tag{19}
$$

By combining Eqns (13,18), the variation of the generalised strain vector can be expressed as,

$$
\delta\mathbf{\Gamma} = \mathbf{N}_{\delta d1}\mathbf{N}_{\delta d2}\,\delta\mathbf{n} \tag{20}
$$

The equilibrium equations are now derived using the Principle of Virtual Work. The internal virtual work is obtained as,

$$\delta W_i = \int_{V_0} \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV = \int_{V_0} \delta \boldsymbol{\Gamma}^T \mathbf{Y}^T \boldsymbol{\sigma}\, dV = \int_{L_0} \delta \boldsymbol{\Gamma}^T \mathbf{D}\, dx \tag{21}$$

where the stress ($\boldsymbol{\sigma}$) and stress resultant (**D**) vectors are defined as,

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma \\ \tau \end{bmatrix} \tag{22}$$

$$\mathbf{D} = \int_{A_0} \mathbf{Y}^T \boldsymbol{\sigma}\, dA = \begin{bmatrix} P & -M_z & M_y & W & B & -T \end{bmatrix}^T \tag{23}$$

The terms of the stress resultant vector (**D**) can be expanded as,

$$\mathbf{D} = \begin{bmatrix} P \\ -M_z \\ M_y \\ W \\ B \\ -T \end{bmatrix} = \begin{bmatrix} \int_{A_0} \sigma\, dA \\ \int_{A_0} y\sigma\, dA \\ \int_{A_0} z\sigma\, dA \\ \int_{A_0} (x^2 + y^2)\sigma\, dA \\ \int_{A_0} \bar{\omega}\sigma\, dA \\ \int_{A_0} 2n\tau\, dA \end{bmatrix} \tag{24}$$

where P, M_z, M_y, W, B and T are the axial force, bending moment about the z-axis, bending moment about the y-axis, Wagner stress resultant, bimoment and uniform torque respectively. By substituting Eqn (20) into Eqn (21), the virtual internal work is obtained as,

$$\delta W_i = \delta \mathbf{n}^T \mathbf{p} \tag{25}$$

where **p** is the internal force in the local system, defined as,

$$\mathbf{p} = \int_{L_0} \mathbf{N}_{\delta d2}^T \mathbf{N}_{\delta d1}^T \mathbf{D}\, dx \tag{26}$$

The virtual external work is defined as,

$$\delta W_e = \delta \mathbf{N}^T \mathbf{Q}_{ext} = \delta \mathbf{n}^T \mathbf{F} \mathbf{Q}_{ext} \tag{27}$$

where $\mathbf{Q}_{ext}$ is the vector of external loads in the directions of the global degrees of freedom, and $\mathbf{F}$ is the transformation matrix, as detailed in (Zhang, X., Rasmussen, K. J. R. and Zhang, H. 2011).

Using the virtual work principle, $\delta W_i = \delta W_e$, the equilibrium equations are obtained by combining Eqns (25,27) and utilising that $\delta\mathbf{n}$ is arbitrary,

$$\mathbf{p} - \mathbf{F}\mathbf{Q}_{ext} = \mathbf{0} \tag{28}$$

The element stiffness matrix is obtained from the variation of Eqn. (28). Alternatively, since $\mathbf{p}$ is derived from the virtual internal work, the element stiffness matrix may be obtained from the variation of the virtual internal work,

$$\delta^2 W_i = \int_{V_0} \delta^2 \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV + \int_{V_0} \delta\boldsymbol{\varepsilon}^T \delta\boldsymbol{\sigma}\, dV \tag{29}$$

The 1st term of Eqn. (29) requires the 2nd variation of the generalised strain vector, obtained by substituting Eqns (10,20) into Eqn. (29) and utilizing Eqn. (23),

$$\int_{V_0} \delta^2 \boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV = \delta\mathbf{n}^T \int_{L_0} \mathbf{N}_{\delta d2}^T \delta\mathbf{N}_{\delta d1}^T \mathbf{D}\, dx = \delta\mathbf{n}^T \int_{L_0} \mathbf{N}_{\delta d2}^T \mathbf{G} \mathbf{N}_{\delta d2}\, dx\, \delta\mathbf{n} \tag{30}$$

where matrix $\mathbf{G}$ is termed the stability matrix in local coordinates and is given by,

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & P & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & P & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & M_z & 0 & 0 \\ 0 & 0 & 0 & M_y & M_z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

The 2nd term of Eqn. (29), $\int_{V_0} \delta\boldsymbol{\varepsilon}^T \delta\boldsymbol{\sigma}\, dV$, requires an expression for the variation of the stress vector ($\delta\boldsymbol{\sigma}$). This is dependent on the material (elastic or inelastic) and on whether the analysis involves local/distortional buckling of the cross-section. In the presence of local/distortional buckling, this term may be developed using the variation of the stress resultant vector ($\delta\mathbf{D}$),

$$\int_{V_0} \delta\boldsymbol{\varepsilon}^T \delta\boldsymbol{\sigma}\, dV = \int_{V_0} \delta\boldsymbol{\Gamma}^T \mathbf{Y}^T \delta\boldsymbol{\sigma}\, dV = \int_{L_0} \delta\boldsymbol{\Gamma}^T \delta\mathbf{D}\, dx \tag{32}$$

where

$$
\begin{aligned}
\delta\mathbf{D} &= \int_{A_0} \mathbf{Y}^T \delta\boldsymbol{\sigma}\, dA \\
&= \delta\left(\int_{A_0} \mathbf{Y}^T \boldsymbol{\sigma}\, dA \right) \\
&= \delta(\mathbf{D}) \\
&= \begin{bmatrix}
\frac{\partial P}{\partial \varepsilon_x} & \frac{\partial P}{\partial(-\kappa_z)} & \frac{\partial P}{\partial \kappa_y} & \frac{\partial P}{\partial \kappa_W} & \frac{\partial P}{\partial \kappa_x} & \frac{\partial P}{\partial(-\varphi')} \\
\frac{\partial(-M_z)}{\partial \varepsilon_x} & \frac{\partial(-M_z)}{\partial(-\kappa_z)} & \frac{\partial(-M_z)}{\partial \kappa_y} & \frac{\partial(-M_z)}{\partial \kappa_W} & \frac{\partial(-M_z)}{\partial \kappa_x} & \frac{\partial(-M_z)}{\partial(-\varphi')} \\
\frac{\partial M_y}{\partial \varepsilon_x} & \frac{\partial M_y}{\partial(-\kappa_z)} & \frac{\partial M_y}{\partial \kappa_y} & \frac{\partial M_y}{\partial \kappa_W} & \frac{\partial M_y}{\partial \kappa_x} & \frac{\partial M_y}{\partial(-\varphi')} \\
\frac{\partial W}{\partial \varepsilon_x} & \frac{\partial W}{\partial(-\kappa_z)} & \frac{\partial W}{\partial \kappa_y} & \frac{\partial W}{\partial \kappa_W} & \frac{\partial W}{\partial \kappa_x} & \frac{\partial W}{\partial(-\varphi')} \\
\frac{\partial B}{\partial \varepsilon_x} & \frac{\partial B}{\partial(-\kappa_z)} & \frac{\partial B}{\partial \kappa_y} & \frac{\partial B}{\partial \kappa_W} & \frac{\partial B}{\partial \kappa_x} & \frac{\partial B}{\partial(-\varphi')} \\
\frac{\partial(-T)}{\partial \varepsilon_x} & \frac{\partial(-T)}{\partial(-\kappa_z)} & \frac{\partial(-T)}{\partial \kappa_y} & \frac{\partial(-T)}{\partial \kappa_W} & \frac{\partial(-T)}{\partial \kappa_x} & \frac{\partial(-T)}{\partial(-\varphi')}
\end{bmatrix}
\begin{bmatrix} \delta\varepsilon \\ -\delta\kappa_z \\ \delta\kappa_y \\ \delta\kappa_W \\ \delta\kappa_x \\ -\delta\varphi' \end{bmatrix} \\
&= \mathbf{S}_t\, \delta\boldsymbol{\Gamma}
\end{aligned}
\tag{33}
$$

The matrix $\mathbf{S}_t$ is termed the tangential rigidity matrix and is defined as,

$$
\mathbf{S}_t = \begin{bmatrix}
(EA)_t & (ES_z)_t & (ES_y)_t & (EI_p)_t & (ES_w)_t & 0 \\
(ES_z)_t & (EI_z)_t & (EI_{yz})_t & (EI_{pz})_t & (EI_{zw})_t & 0 \\
(ES_y)_t & (EI_{yz})_t & (EI_y)_t & (EI_{py})_t & (EI_{yw})_t & 0 \\
(EI_p)_t & (EI_{pz})_t & (EI_{py})_t & (EI_{pp})_t & (EI_{pw})_t & 0 \\
(ES_w)_t & (EI_{zw})_t & (EI_{yw})_t & (EI_{pw})_t & (EI_w)_t & 0 \\
0 & 0 & 0 & 0 & 0 & (GJ)_t
\end{bmatrix}
\tag{34}
$$

where the tangent rigidity terms are given by,

$$
\begin{array}{llll}
\left(EA\right)_t = \dfrac{\partial P}{\partial \varepsilon_x} & \left(ES_z\right)_t = \dfrac{-\partial P}{\partial \kappa_z} = \dfrac{-\partial M_z}{\partial \varepsilon_x} & \left(ES_y\right)_t = \dfrac{\partial P}{\partial \kappa_y} = \dfrac{\partial M_y}{\partial \varepsilon_x} & \left(ES_w\right)_t = \dfrac{\partial P}{\partial \kappa_x} = \dfrac{\partial B}{\partial \varepsilon_x} \\
\left(EI_z\right)_t = \dfrac{\partial M_z}{\partial \kappa_z} & \left(EI_y\right)_t = \dfrac{\partial M_y}{\partial \kappa_y} & \left(EI_w\right)_t = \dfrac{\partial B}{\partial \kappa_x} & \left(EI_{pp}\right)_t = \dfrac{\partial W}{\partial \kappa_W} \\
\left(EI_p\right)_t = \dfrac{\partial P}{\partial \kappa_W} = \dfrac{\partial W}{\partial \varepsilon_x} & \left(EI_{pz}\right)_t = \dfrac{-\partial M_z}{\partial \kappa_W} = \dfrac{-\partial W}{\partial \kappa_z} & \left(EI_{py}\right)_t = \dfrac{\partial M_y}{\partial \kappa_W} = \dfrac{\partial W}{\partial \kappa_y} & \left(EI_{pw}\right)_t = \dfrac{\partial B}{\partial \kappa_W} = \dfrac{\partial W}{\partial \kappa_x} \\
\left(GJ\right)_t \cong GJ & \left(EI_{yz}\right)_t = \dfrac{-\partial M_z}{\partial \kappa_y} = \dfrac{-\partial M_y}{\partial \kappa_z} & \left(EI_{zw}\right)_t = \dfrac{-\partial M_z}{\partial \kappa_x} = \dfrac{-\partial B}{\partial \kappa_z} & \left(EI_{yw}\right)_t = \dfrac{\partial M_y}{\partial \kappa_x} = \dfrac{\partial B}{\partial \kappa_y}
\end{array} \tag{35}
$$

By substituting Eqns (20,33) into Eqn. (32), the term $\int_{V_0} \delta\boldsymbol{\varepsilon}^T \delta\boldsymbol{\sigma}\, dV$ is obtained as,

$$
\int_{V_0} \delta\boldsymbol{\varepsilon}^T \delta\boldsymbol{\sigma}\, dV = \delta\mathbf{n}^T \int_{L_0} \mathbf{N}_{\delta d2}^T \mathbf{N}_{\delta d1}^T \mathbf{S}_t \mathbf{N}_{\delta d1} \mathbf{N}_{\delta d2} dx\, \delta\mathbf{n} \tag{36}
$$

Combining Eqns (29,30,36), the variation of the virtual internal work can be expressed as,

$$
\begin{aligned}
\delta^2 W_i &= \delta\mathbf{n}^T \int_{L_0} \mathbf{N}_{\delta d2}^T \mathbf{G} \mathbf{N}_{\delta d2}\, dx\, \delta\mathbf{n} + \delta\mathbf{n}^T \int_{L_0} \mathbf{N}_{\delta d2}^T \mathbf{N}_{\delta d1}^T \mathbf{S}_t \mathbf{N}_{\delta d1} \mathbf{N}_{\delta d2} dx\, \delta\mathbf{n} \\
&= \delta\mathbf{n}^T \mathbf{K}_l \delta\mathbf{n} \\
&= \delta\mathbf{n}^T \delta\mathbf{p}
\end{aligned} \tag{37}
$$

where, as shown above, δ**p** is the variation of the internal force vector in the local system. The element local stiffness matrix ($\mathbf{K}_l$) is obtained as,

$$
\mathbf{K}_l = \int_{L_0} \mathbf{N}_{\delta d2}^T \mathbf{G} \mathbf{N}_{\delta d2}\, dx + \int_{L_0} \mathbf{N}_{\delta d2}^T \mathbf{N}_{\delta d1}^T \mathbf{S}_t \mathbf{N}_{\delta d1} \mathbf{N}_{\delta d2} dx \tag{38}
$$

5. Calculation of tangent rigidities

The tangent rigidities of the locally/distortionally buckled member can be determined from a non-linear post-local or post-distortional buckling analysis of a length of member as described in (Young, B. and Rasmussen, K. J. R. 1997). The calculated reduced rigidities can subsequently be read as input data to the non-linear beam element analysis program. In the post-local or post-distortional buckling analysis, the length of the member should be chosen to ensure that both local and distortional buckling deformations develop freely, but global buckling is avoided. For sections subject to local buckling only (e.g., an I-section), a length of one local buckling half-wavelength can be chosen, and the applied deformation may be applied by the use of rigid bars (plates) attached to the component plates such that the end displacements vary linearly according to the required deformation. For sections subject to distortional buckling only (e.g., back-to-back lipped channel sections with relatively small lips), the distortional buckling mode is associated with longitudinal displacements which are restrained when applying end displacements corresponding to major or minor axis bending or warping torsion. Consequently, to reduce the influence of such restraint on the distortional buckling deformations, it is convenient to analyse lengths of members that allow multiple distortional buckles, say five, to develop. The analyses of one or numerous local/distortional buckles produce tangent rigidities that are average values for the length of member analysed.

The vector of generalised strains is given by,

$$\mathbf{\Gamma} = \begin{bmatrix} \varepsilon_x \\ -\kappa_z \\ \kappa_y \\ \kappa_W \\ \kappa_x \\ -\varphi' \end{bmatrix} = \begin{bmatrix} u' \\ -v'' \\ -w'' \\ 0 \\ -\varphi'' \\ -\varphi' \end{bmatrix} + \begin{bmatrix} \frac{1}{2}(v')^2 + \frac{1}{2}(w')^2 \\ -\varphi w'' \\ \varphi v'' \\ \frac{1}{2}(\varphi')^2 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

where, for a short length of member, the nonlinear terms in v and w may be ignored. Hence, in the post-local and post-distortional buckling analyses, small increments of generalised stain may be applied as,

$$\Delta\mathbf{\Gamma} = \begin{bmatrix} \Delta\varepsilon_x \\ -\Delta\kappa_z \\ \Delta\kappa_y \\ \Delta\kappa_W \\ \Delta\kappa_x \\ -\Delta\varphi' \end{bmatrix} \cong \begin{bmatrix} \Delta u' \\ -\Delta v'' \\ -\Delta w'' \\ \varphi'\Delta\varphi' \\ -\Delta\varphi'' \\ -\Delta\varphi' \end{bmatrix} \tag{40}$$

Considering Eqns (35,40), the tangent rigidities can be calculated as,

$$\begin{array}{llll}
(EA)_t \cong \dfrac{\Delta P}{\Delta u'} & (ES_z)_t \cong -\dfrac{\Delta M_z}{\Delta u'} \cong -\dfrac{\Delta P}{\Delta v''} & (ES_y)_t \cong \dfrac{\Delta M_y}{\Delta u'} \cong -\dfrac{\Delta P}{\Delta w''} & (ES_w)_t \cong \dfrac{\Delta B}{\Delta u'} \cong -\dfrac{\Delta P}{\Delta \varphi''} \\
(EI_z)_t \cong \dfrac{\Delta M_z}{\Delta v''} & (EI_y)_t \cong -\dfrac{\Delta M_y}{\Delta w''} & (EI_w)_t \cong -\dfrac{\Delta B}{\Delta \varphi''} & (EI_{pp})_t \cong \dfrac{\Delta W}{\varphi'\Delta\varphi'} \\
(EI_p)_t \cong \dfrac{\Delta W}{\Delta u'} \cong \dfrac{\Delta P}{\varphi'\Delta\varphi'} & (EI_{pz})_t \cong -\dfrac{\Delta W}{\Delta v''} \cong -\dfrac{\Delta M_z}{\varphi'\Delta\varphi'} & (EI_{py})_t \cong -\dfrac{\Delta W}{\Delta w''} \cong \dfrac{\Delta M_y}{\varphi'\Delta\varphi'} & (EI_{pw})_t \cong \dfrac{\Delta B}{\varphi'\Delta\varphi'} \cong -\dfrac{\Delta W}{\Delta\varphi''} \\
(GJ)_t \cong GJ & (EI_{yz})_t \cong \dfrac{\Delta M_z}{\Delta w''} \cong -\dfrac{\Delta M_y}{\Delta v''} & (EI_{zw})_t \cong \dfrac{\Delta M_z}{\Delta\varphi''} \cong -\dfrac{\Delta B}{\Delta v''} & (EI_{yw})_t \cong -\dfrac{\Delta M_y}{\Delta\varphi''} \cong -\dfrac{\Delta B}{\Delta w''}
\end{array} \tag{41}$$

The tangent shear modulus G_t is assumed to be equal to the full shear modulus G, so that $(GJ)_t = GJ$. This implies that the uniform torsion rigidity is assumed not to be affected by local or distortional buckling, which has been validated for the case of elastic local buckling (Rasmussen, K. J. R. 2004).

Beam elements subject to bending or combined compression and bending require different combinations of axial straining and curvature to be considered while calculating tangent rigidities. The combinations are here obtained by applying distributions of the longitudinal strain (ε) defined by ξ = -1.0, -0.9,……,1.0 as shown in Figure 4 for major axis bending, where ξ is defined as the ratio of the strain in the bottom flange to the strain in the top flange, such that ξ = -1.0 represents the case of pure compression and ξ = 1.0 represents the case of pure bending. For each combination of bending and compression, the local/distortional buckling half-wavelengths may be obtained from a finite strip analysis program, e.g. THIN-WALL (Papangelis, J. P. and Hancock, G. J. 1995) or CUFSM (Schafer, B. W. and Adany, S. 2006).

In determining the tangent rigidities, a locally/distortionally buckled cell of a certain length was analysed using the geometric nonlinear finite element analysis in (ABAQUS 2009). The cell was subjected to increasing levels of compression, bending and warping, and at each level, small displacements were applied at the ends to produce additional small pure axial strain, curvature about the major and minor axes or warping, as per Eqn. (40). For each additional small generalized strain, the changes of the stress resultants were computed by appropriate integration of the stress over the area. Hence, the tangent rigidities could be obtained as the ratios of the change of stress resultant to the change of generalized strain, as expressed in Eqn. (41). See (Rasmussen, K. J. R. 1997; Young, B. and Rasmussen, K. J. R. 1997) for further details.

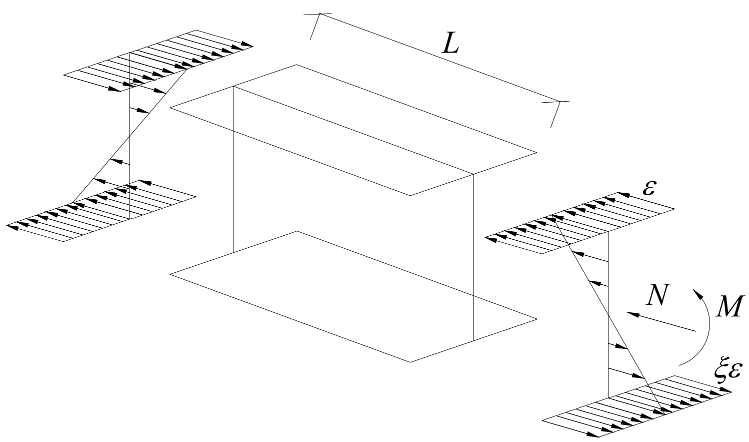


Figure 4: Load combinations defined in strain for major axis bending

6. Incorporation of reduced rigidity into program OpenSees

It is well known that the stiffness of a member reduces as a result of local/distortional buckling of the cross section, which may be expressed generically as:

$$(EI)_t = \tau_g EI \tag{42}$$

where EI and $(EI)_t$ can be any tangent rigidity. When the section is not locally/distortionally buckled, the geometric reduction factor τ_g is unity and $(EI)_t = EI$. A similar expression has been incorporated in AISC specification (AISC 360-10 2010) to take account of stiffness reduction due to member yielding.

The basic principle of the presented method is to find appropriate reduction factors (τ_g) for each tangent rigidity term. In the nonlinear beam element structural analysis, the reduction of tangent rigidities depends on the extent of local/distortional buckling as well as the particular combination of the internal stress resultants (P, M_z, etc.). Thus τ_g is not a constant factor but rather, it may change at each increment throughout the analysis.

Using τ_g-factors, the tangent rigidity matrix can be expressed as,

$$\mathbf{S}_t = \begin{bmatrix} (\tau_g)_{EA} EA & (\tau_g)_{ES_z} ES_z & (\tau_g)_{ES_y} ES_y & (\tau_g)_{EI_p} EI_p & (\tau_g)_{ES_w} ES_w & 0 \\ (\tau_g)_{ES_z} ES_z & (\tau_g)_{EI_z} EI_z & (\tau_g)_{EI_{yz}} EI_{yz} & (\tau_g)_{EI_{pz}} EI_{pz} & (\tau_g)_{EI_{zw}} EI_{zw} & 0 \\ (\tau_g)_{ES_y} ES_y & (\tau_g)_{EI_{yz}} EI_{yz} & (\tau_g)_{EI_y} EI_y & (\tau_g)_{EI_{py}} EI_{py} & (\tau_g)_{EI_{yw}} EI_{yw} & 0 \\ (\tau_g)_{EI_p} EI_p & (\tau_g)_{EI_{pz}} EI_{pz} & (\tau_g)_{EI_{py}} EI_{py} & (\tau_g)_{EI_{pp}} EI_{pp} & (\tau_g)_{EI_{pw}} EI_{pw} & 0 \\ (\tau_g)_{ES_w} ES_w & (\tau_g)_{EI_{zw}} EI_{zw} & (\tau_g)_{EI_{yw}} EI_{yw} & (\tau_g)_{EI_{pw}} EI_{pw} & (\tau_g)_{EI_w} EI_w & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ \end{bmatrix} \tag{43}$$

where the unreduced rigidities can be calculated according to the geometry the cross-sections, and the τ_g -factors are defined as,

$$\begin{aligned} &(\tau_g)_{EA} = \frac{(EA)_t}{EA} \quad (\tau_g)_{ES_z} = \frac{ES_z}{(ES_z)_t} \quad (\tau_g)_{ES_y} = \frac{ES_y}{(ES_y)_t} \quad (\tau_g)_{EI_p} = \frac{EI_p}{(EI_p)_t} \\ &(\tau_g)_{ES_w} = \frac{ES_w}{(ES_w)_t} \quad (\tau_g)_{EI_z} = \frac{EI_z}{(EI_z)_t} \quad (\tau_g)_{EI_{yz}} = \frac{EI_{yz}}{(EI_{yz})_t} \quad (\tau_g)_{EI_{pz}} = \frac{EI_{pz}}{(EI_{pz})_t} \\ &(\tau_g)_{EI_{zw}} = \frac{EI_{zw}}{(EI_{zw})_t} \quad (\tau_g)_{EI_y} = \frac{EI_y}{(EI_y)_t} \quad (\tau_g)_{EI_{py}} = \frac{EI_{py}}{(EI_{py})_t} \quad (\tau_g)_{EI_{yw}} = \frac{EI_{yw}}{(EI_{yw})_t} \\ &(\tau_g)_{EI_{pp}} = \frac{EI_{pp}}{(EI_{pp})_t} \quad (\tau_g)_{EI_{pw}} = \frac{EI_{pw}}{(EI_{pw})_t} \quad (\tau_g)_{EI_w} = \frac{EI_w}{(EI_w)_t} \end{aligned} \tag{44}$$

For members with doubly-symmetric cross-sections subject to pure compression, due to the symmetry of the cross-section and the local/distortional buckling mode, only the diagonal terms of the tangent rigidity matrix (Eqn. (34)) and $(EI_p)_t$ are non-zero during the analysis. Thus the matrix simplifies to,

$$\mathbf{S}_t = \begin{bmatrix} (\tau_g)_{EA}(EA) & 0 & 0 & (\tau_g)_{EI_p}(EI_p) & 0 & 0 \\ 0 & (\tau_g)_{EI_z}(EI_z) & 0 & 0 & 0 & 0 \\ 0 & 0 & (\tau_g)_{EI_y}(EI_z) & 0 & 0 & 0 \\ (\tau_g)_{EI_p}(EI_p) & 0 & 0 & (\tau_g)_{EI_{pp}}(EI_{pp}) & 0 & 0 \\ 0 & 0 & 0 & 0 & (\tau_g)_{EI_w}(EI_w) & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ \end{bmatrix} \quad (45)$$

For doubly-symmetric cross-sections subject to pure bending or combined compression and bending, other off-diagonal terms than $(EI_p)_t$ become non-zero as local/distortional buckling deformations develop. The effects of these terms were checked by switching on and off each term separately in the tangent rigidity matrix (**S**$_t$) during the verification analyses. It has been verified that for a doubly-symmetric cross-section in bending or combined compression and bending, in addition to the $(EI_p)_t$ term, the off-diagonal $(EI_{yw})_t$ must also be considered. The tangent rigidity matrix simplifies to,

$$\mathbf{S}_t = \begin{bmatrix} (\tau_g)_{EA}EA & 0 & 0 & (\tau_g)_{EI_p}EI_p & 0 & 0 \\ 0 & (\tau_g)_{EI_z}EI_z & 0 & 0 & 0 & 0 \\ 0 & 0 & (\tau_g)_{EI_y}EI_y & 0 & (\tau_g)_{EI_{yw}}EI_{yw} & 0 \\ (\tau_g)_{EI_p}EI_p & 0 & 0 & (\tau_g)_{EI_{pp}}EI_{pp} & 0 & 0 \\ 0 & 0 & (\tau_g)_{EI_{yw}}EI_{yw} & 0 & (\tau_g)_{EI_w}EI_w & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ \end{bmatrix} \quad (46)$$

Figure 5 explains why the $(EI_{yw})_t$-term develops when local/distortional buckling deformations occur. As per Eqn. (41), the tangent rigidity $(EI_{yw})_t$ is the change of minor axis bending moment (ΔM_y), that occurs as a result of a change of twist strain ($-\Delta\varphi''$), divided by $-\Delta\varphi''$. (It is also the change of bimoment (ΔB), that occurs as a result of a change of minor axis curvature ($-\Delta w''$), divided by $-\Delta w''$). Since $\Delta M_y = \int \Delta\sigma z dA$, it is clear from Figure 5 that once the section is locally/distortionally buckled, the change of longitudinal stress ($\Delta\sigma$) caused by a small additional warping strain ($-\Delta\varphi''$) is uneven in the two flanges and hence, a minor axis bending moment (ΔM_y) is produced. This implies that the $(EI_{yw})_t$-term is non-zero. As will be shown in Section 6, the term can have a substantial influence on the structural response.

The element local stiffness matrix (**K**$_l$) can be obtained from Eqn. (38) by substitution of Eqn. (45) or (46) for **S**$_t$.

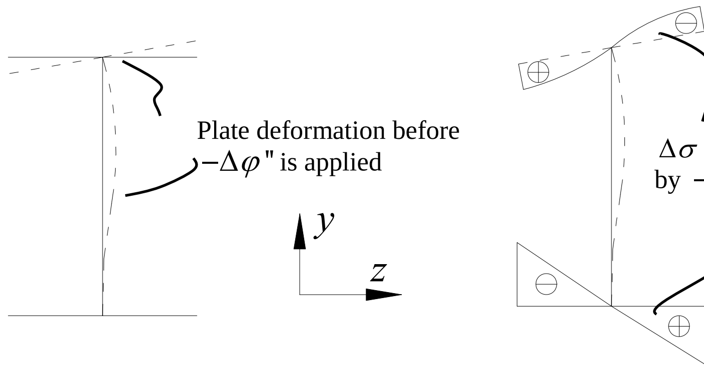


Figure 5: Plate deformation under bending and stresses caused by $-\Delta\varphi''$

On completion of an iteration, the increment in stress resultants ($\delta\mathbf{D}$) needs to be obtained. In the OpenSees framework, increments in stress resultants are obtained by integration of increments in stress over the cross-section as per Eqn. (24). Thus, to comply with the Opensees framework, expressions for the stress increments ($\delta\sigma$) are here developed.

Eqn. (33) may be rewritten as,

$$\delta\mathbf{D} = \mathbf{S}_t\delta\mathbf{\Gamma}$$
$$= \begin{bmatrix} (\tau_g)_{EA}EA & 0 & 0 & (\tau_g)_{EI_p}EI_p & 0 & 0 \\ 0 & (\tau_g)_{EI_z}EI_z & 0 & 0 & 0 & 0 \\ 0 & 0 & (\tau_g)_{EI_y}EI_y & 0 & (\tau_g)_{EI_{yw}}EI_{yw} & 0 \\ (\tau_g)_{EI_p}EI_p & 0 & 0 & (\tau_g)_{EI_{pp}}EI_{pp} & 0 & 0 \\ 0 & 0 & (\tau_g)_{EI_{yw}}EI_{yw} & 0 & (\tau_g)_{EI_w}EI_w & 0 \\ 0 & 0 & 0 & 0 & 0 & (\tau_g)_{GJ}GJ \end{bmatrix} \begin{bmatrix} \delta\varepsilon_x \\ -\delta\kappa_z \\ \delta\kappa_y \\ \delta\kappa_W \\ \delta\kappa_x \\ -\delta\varphi' \end{bmatrix}$$
$$= \begin{bmatrix} (\tau_g)_{EA}EA\delta\varepsilon_x + (\tau_g)_{EI_p}EI_p\delta\kappa_W \\ -(\tau_g)_{EI_z}EI_z\delta\kappa_z \\ (\tau_g)_{EI_y}EI_y\delta\kappa_y + (\tau_g)_{EI_{yw}}EI_{yw}\delta\kappa_x \\ (\tau_g)_{EI_p}EI_p\delta\varepsilon_x + (\tau_g)_{EI_{pp}}EI_{pp}\delta\kappa_W \\ (\tau_g)_{EI_{yw}}EI_{yw}\delta\kappa_y + (\tau_g)_{EI_w}EI_w\delta\kappa_x \\ -GJ\delta\varphi' \end{bmatrix} \tag{47}$$

By using the definition of each term of the cross-section rigidities

$$\begin{aligned} &EA = \int_{A_0} E\,dA && EI_y = \int_{A_0} Ez^2\,dA && EI_z = \int_{A_0} Ey^2\,dA && EI_w = \int_{A_0} E\overline{\omega}^2\,dA \\ &EI_p = \int_{A_0} E\left(x^2+y^2\right)dA && EI_{pp} = \int_{A_0} E\left(x^2+y^2\right)^2 dA && EI_{yw} = \int_{A_0} Ez\,\overline{\omega}dA && GJ = \int_{A_0} G4n^2\,dA \end{aligned} \tag{48}$$

Eqn. (47) may be rewritten as,

$$\delta\mathbf{D} = \mathbf{S}_t\delta\Gamma = \begin{bmatrix} \delta P \\ -\delta M_z \\ \delta M_y \\ \delta W \\ \delta B \\ -\delta T \end{bmatrix} = \begin{bmatrix} \int_{A_0} (\tau_g)_{EA}\delta\varepsilon_x E\,dA + \int_{A_0} (\tau_g)_{EI_p}\delta\kappa_W E(y^2+z^2)\,dA \\ \int_{A_0} -(\tau_g)_{EI_z}\delta\kappa_z E\,y^2 dA \\ \int_{A_0} (\tau_g)_{EI_y}\delta\kappa_y E\,z^2 dA + \int_{A_0} (\tau_g)_{EI_{yw}}\delta\kappa_x E\,z\varpi dA \\ \int_{A_0} (\tau_g)_{EI_p}\delta\varepsilon_x E(y^2+z^2)dA + \int_{A_0} (\tau_g)_{EI_{pp}}\delta\kappa_W E(y^2+z^2)^2 dA \\ \int_{A_0} (\tau_g)_{EI_{yw}}\delta\kappa_y Ez\varpi dA + \int_{A_0} (\tau_g)_{EI_w}\delta\kappa_x E\varpi^2 dA \\ \int_{A_0} -\delta\varphi' G4n^2 dA \end{bmatrix}$$

$$= \begin{bmatrix} \int_{A_0} (\delta\sigma_{p1} + \delta\sigma_{p2})\,dA \\ \int_{A_0} \delta\sigma_{M_z}\, ydA \\ \int_{A_0} (\delta\sigma_{M_{y1}} + \delta\sigma_{M_{y2}})zdA \\ \int_{A_0} (\delta\sigma_{W1} + \delta\sigma_{W2})(y^2+z^2)dA \\ \int_{A_0} (\delta\sigma_{B1} + \delta\sigma_{B2})\varpi dA \\ \int_{A_0} 2n\delta\tau dA \end{bmatrix} \quad (49)$$

where,

$$\begin{array}{llll} \delta\sigma_{p1} = (\tau_g)_{EA}E\delta\varepsilon_x & \delta\sigma_{p2} = (\tau_g)_{EI_p}E(y^2+z^2)\delta\kappa_W & \delta\sigma_{M_z} = -(\tau_g)_{EI_z}Ey\delta\kappa_z & \delta\sigma_{M_{y1}} = (\tau_g)_{EI_z}Ez\delta\kappa_z \\ \delta\sigma_{M_{y2}} = (\tau_g)_{EI_{yw}}E\varpi\delta\kappa_x & \delta\sigma_{W1} = (\tau_g)_{EI_p}E\delta\varepsilon_x & \delta\sigma_{W2} = (\tau_g)_{EI_{pp}}E\delta\kappa_W(y^2+z^2) & \delta\sigma_{B1} = (\tau_g)_{EI_{yw}}E\delta\kappa_y z \\ \delta\sigma_{B2} = (\tau_g)_{EI_w}E\delta\kappa_x\varpi & \delta\tau = -G2n\delta\varphi' & & \end{array} \quad (50)$$

The reduction factors are incorporated into OpenSees by means of multi-dimensional arrays. In general, the tangent rigidities ($(EA)_t$, $(ES_z)_t$, $(EI_z)_t$, etc) are functions of the generalised strains $\varepsilon_x, \kappa_z, \kappa_y, \kappa_x$), which define the deformed state of the locally/distortionally buckled length of member. In elastic analyses, the process is to systematically subject the locally/distortionally buckled cell(s) to increasing levels of one generalised strain, while keeping the other generalised strains constant, and at each combination of generalised strains, apply additional small generalised strain and calculate the tangent rigidities as per Eqn (41), as explained in Section 4. The tangent rigidities are thereby determined for combinations of discrete values of generalised strains. Provided the generalised strains are finely spaced, particularly near the strain combinations causing local/distortional buckling, the tangent rigidities may be obtained by linear interpolation between the discrete values stored in the multi-dimensional arrays.

Thus the tangent rigidities are to be calculated for a large number of combinations of generalised strains. Depending on the particular application and required level of accuracy, they may be assumed to be functions of less than the full set of generalised strains ($\varepsilon_x, \kappa_z, \kappa_y, \kappa_x$), as discussed further in Section 6. For the presented method to be useful in engineering practice, it is also necessary to derive simple expressions for the reduction factors (τ_g) that can be implemented directly into the global analysis without the need for determining the factors *a priori* by a separate analysis of a locally/distortionally buckled cell(s). However, this latter step is beyond the purpose of the present paper which aims to show that the general method is accurate.

Note that the Wagner strain (κ_W) is not considered as a generalised strain in determining tangent rigidities in the present analysis. A separate study was carried out on locally buckled doubly-symmetric beams in pure bending and locally buckled doubly-symmetric columns in pure compression by varying the tangent rigidities associated with the Wagner strain ($(EI_p)_t$ and $(EI_{pp})_t$) from their full unreduced values (EI_p, EI_{pp}) to zero. It was found that the global response was insensitive to the values of $(EI_p)_t$ and $(EI_{pp})_t$ and hence, no attempt was made to determine the rigidities rigorously by subjecting the locally buckled cell(s) to twist rotation (φ') in addition to generalised strains ($\varepsilon_x, \kappa_z, \kappa_y, \kappa_x$) generated by longitudinal displacements. For convenience, in this paper, the reduction factors $(\tau_g)_{EI_p}$ and $(\tau_g)_{EI_{pp}}$ were set equal to $(\tau_g)_{EA}$.

7. Comparison of beam and shell element analyses

The proposed beam element method was verified by comparison with shell finite element analysis results of locally buckled beams under pure bending, which were obtained using the commercial software ABAQUS. Only elastic analyses were performed. An I-section was chosen for the analyses, the geometry of which is shown in Figure 6. The beams were simply supported at both ends. Warping and twisting deformations were restrained at the ends while flexural rotations were free to occur.

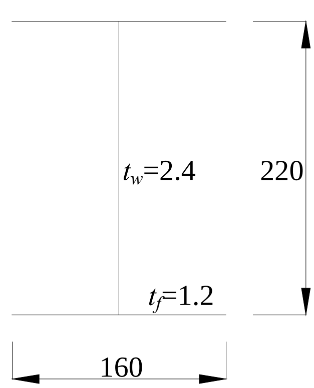


Figure 6: Geometry of the cross-section (dimensions in mm)

For the presented method, the modified Displacement Based Beam-Column Warping Element as introduced in (Zhang, X., Rasmussen, K. J. R. and Zhang, H. 2012) was employed, using a total of twenty elements along the member. In the ABAQUS model, rigid plates were attached to both ends of the member. Contact between the member ends and the rigid plates was modelled by means of node-to-surface contact pairs. A reference node, which was used to apply load and boundary conditions, was defined for each rigid plate.

The member lengths were chosen to be 4 m and 5 m, producing corresponding slenderness ratios (L_{ey}/r_y) of 133 and 167, respectively. An overall imperfection was superimposed onto the perfect geometry in the shape of the elastic overall buckling mode for flexural buckling about the *y*-axis. The magnitude of the imperfection at mid-length was chosen to be 1 mm. The material was assumed to be elastic. The Young's modulus and Poisson's ratio were 200000 MPa and 0.3 respectively.

Because the beams were subjected to uniform bending, the main deformation causing local buckling was major axis curvature (κ_z). The tangent rigidities ($(EA)_t, (EI_z)_t, (EI_y)_t, (EI_w)_t$) were therefore assumed to be functions of only ε_x and κ_z, where the axial strain (ε_x) had to be considered to maintain zero axial force (P=0) after the development of local buckling. However, the tangent rigidity $(EI_{yw})_t$, which encapsulates the coupling between minor axis bending and warping, is primarily associated with flexural-torsional buckling deformations, and so was assumed to be a function of κ_z, κ_y and κ_x. An imperfection in the shape of the local buckling mode with magnitude 0.012 mm (0.01 t_f) was introduced in the analysis determining tangent rigidities.

Figures 7-10 plot the applied moment versus in-plane deflection and applied moment versus out-of-plane deflection for the 4 meter and 5 meter beams. For the 5 meter beam, two different cases were considered in the presented method. In case 1, the off-diagonal term $(EI_{yw})_t$ is ignored while assembling the stiffness matrix, whereas in case 2, the term $(EI_{yw})_t$ is considered, as shown in Figures 7 and 8. It can be seen from Figures 7 and 8 that there are significant discrepancies between the load-deflection curves for case 1 and ABAQUS. The discrepancies demonstrate that for a given applied moment, the stress resultants M_y and B may be significantly underestimated after the occurrence of local buckling when the $(EI_{yw})_t$ term is ignored. The results from case 2 show good agreement with the ABAQUS results. The discrepancies are within 5 percent for the in-plane deflection curves throughout the analyses. For the out-of-plane deflection curves, good agreement is achieved near the ultimate load, whereas small differences can be observed during initial lateral buckling.

It can be concluded from the results shown in Figures 7-10 that unlike beams with compact sections which laterally buckle suddenly at the flexural-torsional buckling load, the lateral buckling of the beams with slender sections develops gradually after reaching the local/distortional buckling load. This phenomenon can be explained by reference to the $(EI_{yw})_t$ term. As shown in Figure 11, which illustrates the relationship between $(EI_{yw})_t$ and major axis curvature (v''), $(EI_{yw})_t$ is triggered soon after the occurrence of local buckling, and rapidly assumes values that are comparable to EI_y and EI_w. The presence of the $(EI_{yw})_t$ term reduces the section's resistance to flexural-torsional buckling and leads to a more rapid growth of lateral deflections for a given applied moment.

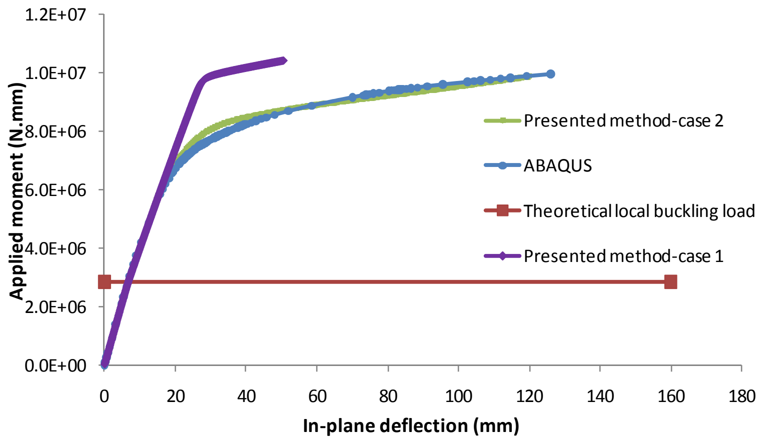


Figure 7: Moment in-plane deflection (5 m)

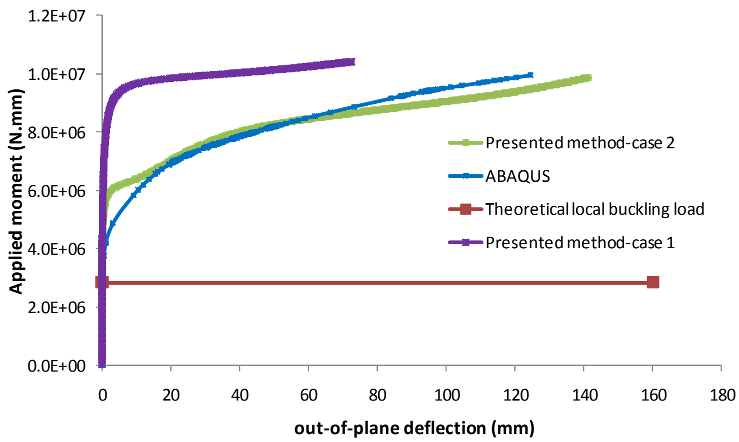


Figure 8: Moment out-of-plane deflection (5 m)

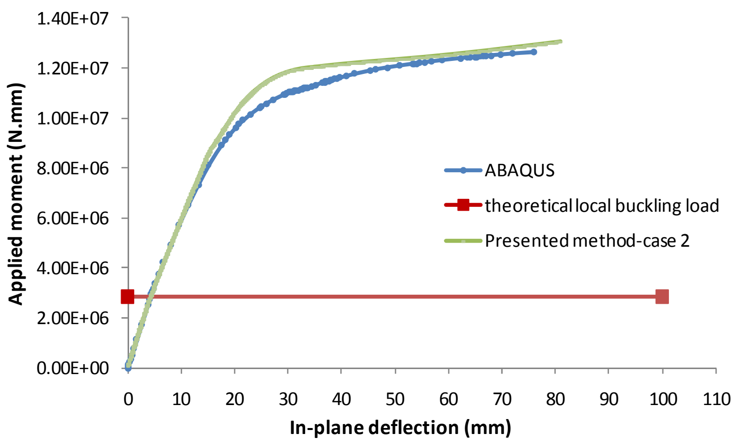


Figure 9: Moment in-plane deflection (4 m)

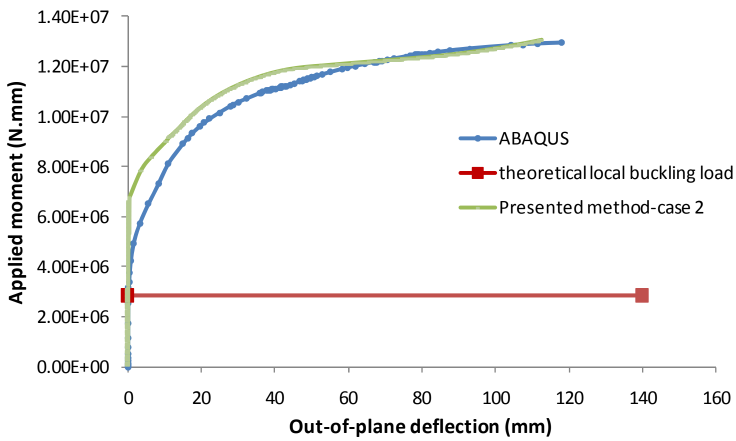


Figure 10: Moment out-of-plane deflection (4 m)

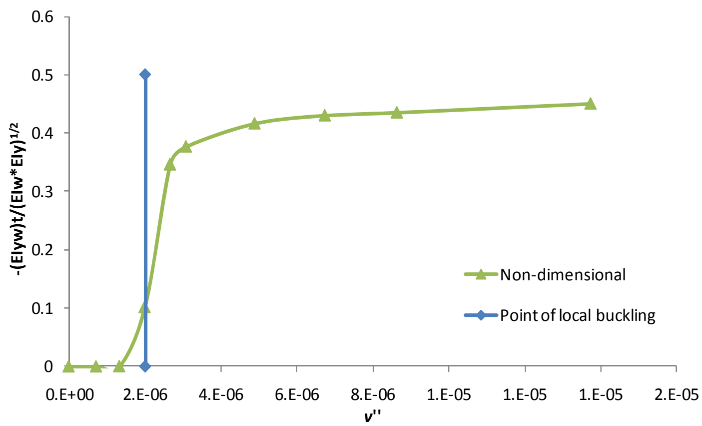


Figure 11: Relations between $(EI_{yw})_t$ and v''

8. Conclusions

A nonlinear analysis method for locally/distortionally buckled members has been presented and applied to doubly symmetric sections. The method allows localized cross-sectional deformations such as local and distortional buckling to be captured in simple beam-element-based nonlinear analyses, thus enabling non-compact or slender sections to be modeled without discretization of the cross-section in shell or plate finite elements. The method has been verified by comparison with ABAQUS shell element results of simply supported beams under pure bending. Good agreement was achieved for the load-deflection curves. The necessity of including off-diagonal terms in the analysis while calculating out-of-plane flexural-torsional buckling displacements was also presented and explained.

The paper demonstrates how the effect of local/distortional buckling can be captured in the beam-element analysis by using tangent rigidities in place of full section rigidities (e.g. $(EI_z)_t$ in place of EI_z) in calculating the element stiffness matrix. The determination of tangent rigidities is discussed in detail in the paper as are approximate ways of determining the rigidities. While the paper shows that the response of locally buckled beams can be accurately predicted using beam elements with reduced tangent rigidities, the ultimate aim of the presented analysis is to produce a tool for practicing engineers that feature simple expressions for the reduction of the section rigidities caused by local/distortional buckling in a similar way to the τ-factors currently specified in the AISC-360 Specification to account for the reduction in rigidity caused by yielding. Research is ongoing to determine such simplified expressions.

References

ABAQUS (2009). Inc. ABAQUS. Ver. 6.8. analysis user's manual.

AISC 360-10 (2010). "Specification for Structural Steel Buildings " American Institute of Steel Construction (AISC), Chicago, Illinois.

Alemdar, B. N. (2001). "Distributed plasticity analysis of steel building structural systems." PhD thesis, Georgia Institute of Technology.

Mckenna, F. T. (1997). "Object-oriented finite element programming: Frameworks for analysis, algorithms and parallel computing". PhD thesis, University of California, Berkeley.

Papangelis, J. P. and Hancock, G. J. (1995). "Computer analysis of thin-walled structural members." Computers & Structures **56**(1): 157-176.

Pi, Y. L., Trahair, N. S. and Rajasekaran, S. (1992). "Energy equation for beam lateral buckling." Journal of Structural Engineering **118**(6): 1462-1479.

Rasmussen, K. J. R. (1997). "Bifurcation of locally buckled members." Thin-Walled Structures **28**(2): 117-154.

Rasmussen, K. J. R. (2004). "Torsional rigidity of locally buckled sections", Proceedings of the Fourth International Conference on Coupled Instabilities in Metal Structures. Rome, Italy, Eds. M. Pignataro**:** 57-65.

Rasmussen, K. J. R. and Hasham, A. S. (1997). "Flexural and flexural-torsional bifurcation of locally buckled beam-columns." Thin-Walled Structures **29**(1-4): 203-233.

Schafer, B. W. and Adany, S. (2006). "Buckling analysis of cold-formed steel members using CUFSM: Conventional and constrained finite strip methods". 18th International Specialty Conference on Cold-formed Steel Structures: Recent Research and Developments in Cold-formed Steel Design and Construction.**:** 39-54.

Young, B. and Rasmussen, K. J. R. (1997). "Bifurcation of singly symmetric columns." Thin-Walled Structures **28**(2): 155-177.

Zhang, X., Rasmussen, K. J. R. and Zhang, H. (2011). "Formulation and implementation of three-dimensional doubly symmetric beam-column analyses with warping effect in OpenSees", Research Report R917, School of Civil Engineering, Centre for Advanced Structural Engineering, University of Sydney.

Zhang, X., Rasmussen, K. J. R. and Zhang, H. (2012). "Formulation and implementation of three-dimensional doubly symmetric beam-column analyses with warping effect in OpenSees". The Seventh International Conference on Advances in Steel Structures. Nanjing, China.

Ziemian, R. D. (1990). "Advanced methods of inelastic analysis in the limit states design of steel structures." PhD thesis, Cornell University.