GBT Buckling Analysis of Generally Loaded Thin-Walled Members Exhibiting Arbitrary Flat-Walled Cross-Sections

Rui Bebiano¹, Cilmar Basaglia², Dinar Camotim³, Rodrigo Gonçalves⁴

Abstract

A recently developed Generalized Beam Theory (GBT) formulation (Basaglia et al. 2013) is employed to perform buckling analyses of thin-walled members (i) exhibiting arbitrary flat-walled cross-sections, namely combinations of closed cells and open branches, and (ii) subjected to general loadings, including transverse loads acting away from the shear center axis. The above formulation, which is briefly presented in the first part of the paper, is employed to analyze the buckling behavior of two closed-cell beams namely (i) a RHS cantilever acted by two tip point loads and (ii) a closed-flange I-section simply supported beam subjected to a uniformly distributed load. In both cases, the loads are applied at the shear center axis and also at a parallel axes located at the top and bottom of the beam. The results presented and discussed, which consist of pre-buckling stress fields, buckling curves and buckling mode configurations, are obtained by means of the newly released code GBTUL 2.0 and validated by means of shell finite element analyses carried out in the code ANSYS.

1. Introduction

Evaluating the structural efficiency of thin-walled steel members usually requires assessing their linear buckling behavior, a task that involves determining the relevant buckling modes and the associated bifurcation loads/stresses. However, the difficulty in performing this task tends to increase with (i) the cross-section shape complexity and (ii) the slenderness of its walls, which brings up buckling modes exhibiting combinations of local, distortional, shear and wall transverse extension deformations – in members with cross-sections containing closed cells, it is also well-known that torsion is associated with non-null shear strains stemming from the presence of cell shear flows (Murray 1984).

Generalized Beam Theory (GBT) is a thin-walled bar theory that takes into account all the various deformation types mentioned in the previous paragraph – thus, it provides an efficient alternative to the shell finite element (SFE) or constrained finite strip (cFSM) methods. Originally proposed by Schardt (1989), GBT has been considerably developed in the last decade, mostly due to the research

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efforts carried out at the University of Lisbon (e.g., Camotim et al. 2010a, b), where it has been employed in the context of a wide variety of structural problems, involving the linear, buckling, vibration, post-buckling and dynamic behaviors of thin-walled members and structural systems, namely trusses and frames. The main distinguishing feature and advantage of the GBT approach is the fact that it expresses the cross-section deformation as a linear combination of contributions from structurally meaningful cross-section deformation modes. Each “contribution” consists of a product involving (i) a shape function defined along the cross-section mid-line (the deformation mode profile) and (ii) the corresponding longitudinal amplitude function, providing the variation of its amplitude along the member length. The procedures leading to the determination of the each of the above shape and amplitude functions constitute cross-section analysis and member analysis, respectively. Finally, note that GBT retains the simplicity of one-dimensional theories, while exhibiting “SFE-like” capabilities.

Recently, Basaglia & Camotim (2013) significantly extended the domain of application of the GBT buckling analyses, by developing, numerical implementing and validating a formulation capable of handling arbitrary loadings, including transverse loads acting away from the member shear center axis. However, the above formulation was developed and validated exclusively in the context of open-section thin-walled members, i.e., did not cover members with cross-sections containing closed cells. On the other hand, a novel cross-section analysis procedure capable of handling efficiently arbitrary flat-walled cross-sections was recently developed (Gonçalves et al. 2014, Bebiano et al. 2015a)\(^5\).

Therefore, the objective of this work is to address the application of the GBT formulation developed by Basaglia & Camotim (2013) to members with cross-sections containing closed cells, and to assess the quality of the buckling results provided by such application. It is worth noting that the main specific feature of the above members is the possible need to include in the GBT analysis cell shear flow deformation modes (e.g., the inclusion of one particular cell shear flow deformation mode is indispensable to capture adequately the member torsional behavior). The GBT-results presented and discussed in this work are obtained with the code GBTul 2.0, which is based on GBT, was developed at the University of Lisbon and has been very recently made available online (Bebiano et al. 2014, 2015b). These results concern single and two-cell beams subjected to transverse loads acting on different locations with respect to the beam shear center axis. Moreover, for validation purposes, some of the GBT-based results are compared with values obtained by means of accurate shell finite element simulations performed in the commercial code ANSYS (SAS 2009).

2. Generalized Beam Theory – Brief Overview

As mentioned above, GBT is a one-dimensional bar theory that expresses/discretizes the member deformed configuration as a linear combination of cross-section deformation modes multiplied by their (modal) amplitude functions. A very brief overview of GBT is presented next – more detailed accounts can be found in the literature (e.g., Camotim et al. 2010a, b).

Consider the prismatic thin-walled member with the (supposedly arbitrary) cross-section depicted in Fig. 1(a), in which local coordinate systems \(x – s – z\) are adopted at each wall, as shown in Fig. 1(b). In GBT, the wall mid-plane axial, transverse and normal displacement components – \(u(x, s)\), \(v(x, s)\) and \(w(x, s)\) – are given by

\(^5\) It is worth noting that a GBT cross-section analysis procedure capable of handling arbitrary flat-walled cross-sections was previously available (Gonçalves et al. 2010). However, this procedure could not be easily automated and, moreover, involved a different deformation mode classification and hierarchy.
\[ u(x, s) = u_k(s)\varphi_{k,x}(x) \quad v(x, s) = v_k(s)\varphi_k(x) \quad w(x, s) = w_k(s)\varphi_k(x) \]

where (i) \(u_k(s), v_k(s)\) and \(w_k(s)\) are the mid-line functions defining cross-section deformation mode \(k\) (or “GBT mode \(k\)”) (ii) \(\varphi_k(x)\) or \(\varphi_{k,x}(x)\) is the amplitude function describing its variation along the member length, and (iii) \(1 \leq k \leq N_d\), where \(N_d\) is the total number of deformation modes. Thus, the member deformed configuration can be expressed as a sum of contributions from the \(N_d\) deformation modes – the contribution of mode \(k\) is the product of its mid-line displacement functions by the corresponding amplitude function. Alternatively, (1) can be written in matrix form as

\[ u = u^T \varphi_{x} \quad v = v^T \phi \quad w = w^T \phi \]

where (i) \(u, v\) and \(w\) are vectors containing the \(u_k(s)\), \(v_k(s)\) and \(w_k(s)\) functions, respectively, and (ii) \(\phi\) is a vector containing the corresponding amplitude functions \(\varphi_k(x)\).

Figure 1: (a) Prismatic thin-walled member with a supposedly arbitrary cross-section and global coordinate system, and (b) infinitesimal wall element with its local coordinate system and displacement components.

The member elastic strain energy \(U\) is given by (\(V\) is the member volume)

\[ U = \frac{1}{2} \int_V \sigma_{ij} \varepsilon_{ij} dV \]

where \(\sigma_{ij}\) and \(\varepsilon_{ij}\) are the various stress and strain components included in the analysis – after adopting the Kirchhoff-Love hypotheses, a plane stress state, with components \(\sigma_{xx}, \sigma_{ss}\) and \(\tau_{xs}\), takes place in the member is mid-plane. Moreover, by using Eq. (2) and considering Cauchy’s linear strain tensor and Hooke’s constitutive law (Bebiano et al. 2015a), Eq. (3) can be expressed in the form

\[ U = \frac{1}{2} \int_L \left( \varphi_{xx}^T C \varphi_{xx} + \varphi_{x}^T D \varphi_{x} + \varphi^T B \phi + \varphi_{xx}^T E \phi \right) dx \]

where \(L\) is the member length and \(C, B, D\) and \(E\) are \(N_d \times N_d\) linear stiffness matrices, associated with various cross-section mechanical properties, namely (i) primary/secondary warping, (ii) transverse extension/flexure, (iii) plate distortion/torsion and (iv) membrane/flexural Poisson effects, respectively – the analytical expressions for their components are provided in Appendix A.

The first step of a GBT structural analysis is the determination of the cross-section deformation modes \((u_k(s), v_k(s)\) and \(w_k(s)\) functions) and associated mechanical properties \((C_{ik}, B_{ik}, D_{ik}\) and \(E_{ik}\) components), which is done through a systematic procedure termed Cross-Section Analysis. In this work, a recently developed version of this procedure (Gonçalves et al. 2014, Bebiano et al. 2015a), applicable to arbitrary flat-walled members. Its first step consists of specifying a cross-section nodal discretization and automatically computing the \(N_d\) deformation modes, which may be grouped into 3 main families: (i) the Vlasov modes (for which \(\gamma_{xs} = \varepsilon_{ss} = 0\)), (ii) the Shear modes (for which \(\gamma_{xs} \neq 0; \varepsilon_{ss} = 0\)
and (iii) the Transverse Extension modes (for which $\varepsilon_{ss} \neq 0$) – all of them can be further divided into several deformation mode sub-families (Bebiano et al. 2015a). At this point, it is possible to select any subset of $n_d$ deformation modes ($1 \leq n_d \leq N_d$) to be included in the GBT analysis, thus reducing the number of degrees of freedom involved.

After knowing the cross-section deformation modes and modal mechanical properties, it becomes possible to perform the Member Analysis, which provides the solution of the buckling eigenvalue problem under consideration: buckling loads/moments (eigenvalues) and associated mode shapes (eigenvectors), defined in terms of the $\phi_k(x)$ functions. Whenever the loading (or, more precisely, the associated applied stress diagrams) is simple enough, it is a straightforward matter to determine the geometrical stiffness terms required to perform a member linear buckling analysis (e.g., Bebiano et al. 2007). However, for more general/complex loadings (e.g., transverse loads acting away from the member shear center axis) such an approach is not viable – instead, it is indispensable to determine the applied (pre-buckling) beforehand, by means of a preliminary first-order pre-buckling analysis.

2.1 Pre-Buckling Analysis

The first-order equilibrium equations may be established by means of the Principle of Stationary Potential Energy, which reads

$$\delta(U + \Pi) = 0$$

(5)

where $\Pi$ is the potential of the applied loads. In order to define this potential, let $Q_s$ be a generic loading consisting of surface distributed loads applied on a member wall mid-plane and acting along the $s$ direction – in this case, it is assumed that they can be expressed in the form

$$Q_s(x, s) = q_s(s)\phi(x)$$

(6)

where $q_s(s)$ describes the loading profile along the cross-section mid-line and $\phi(x)$ is the corresponding longitudinal amplitude function. Then, the potential of this applied loading can be expressed as

$$\Pi = -\int_M q_s\phi v ds dx$$

(7)

where $M$ stands for the whole member mid-plane domain. Introducing (2) into (7) and integrating along the cross-section mid-line $S$, one obtains

$$\Pi = -\int_S \phi^T q \phi dx$$

(8)

where the first-order tensor $q$ components are generalized modal forces, given by

$$q_i = \int_S q_s v_i ds$$

(9)

Figure 2 shows two illustrative cross-section loading profiles $q_s(s)$, consisting of loads acting transversally on a lipped channel (horizontal) web: (i) an uniformly distributed load $p$ (Fig. 2(a)) and (ii) a concentrated load $P$ (Fig. 2(b)) $-H(s)$ and $\delta(s)$ are the Heaviside and Dirac delta functions). Finally, introducing (4) and (8) into (5), one is led to the weak form of the member first-order equilibrium equations, which reads

$$\int_L \left( \delta \phi_{xx}^T C \phi_{xx} + \delta \phi_x^T D \phi_x + \delta \phi^T B \phi + \delta \phi_{xx}^T E \phi + \delta \phi^T E^T \phi_{xx} - \delta \phi^T \phi \phi \right) dx = 0$$

(10)

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6 Obviously, the number, nature and quality of the deformation modes obtained depend on the cross-section nodal discretization, involving (i) natural internal nodes, (ii) natural end nodes and (iii) intermediate (user-defined) nodes (Bebiano et al. 2015a).
Stationary Potential

In linear buckling analysis, the equilibrium equations may determine the above stress field and its solution \( \mathbf{q}^0 \) (vector whose components are the pre-buckling modal amplitude functions \( \varphi_j^0(x) \)), defines the pre-buckling deformed configuration, given by

\[
\begin{align*}
u^0 &= \mathbf{v}^T \varphi^0 \\
w^0 &= \mathbf{w}^T \varphi^0
\end{align*}
\] (11)

As for the membrane components of the pre-buckling stresses, they are provided by the expressions

\[
\begin{align*}
\sigma_{xx}^0 &= \frac{E}{1-\nu^2} \mathbf{u}^T \varphi^0_{,x} + \frac{E\nu}{1-\nu^2} \mathbf{v}^T \varphi^0_{,s} \\
\sigma_{ss}^0 &= \frac{E}{1-\nu^2} \mathbf{v}^T \varphi^0_{,s} + \frac{E\nu}{1-\nu^2} \mathbf{u}^T \varphi^0_{,x} \\
\tau_{xs}^0 &= G \left( \mathbf{u}^T \varphi^0_{,s} + \mathbf{v}^T \varphi^0_{,x} \right)
\end{align*}
\] (12a)

where \( E, G, \nu \) are the material Young’s modulus, shear modulus and Poisson’s ratio.

Finally, note that, although it is obviously more accurate to use all the \( N_d \) GBT deformation modes to determine the above stress fields, it also possible to achieve quite good results by considering only a given subset of them (\( n_d^0 \leq N_d \)) – the pre-buckling stress field thus combines only some of specific patterns.

2.2 Buckling Analysis

In linear buckling analyses, the equilibrium equations may also be established by means of the Principle of Stationary Potential Energy, which now reads

\[
\delta(U + U^0) = 0
\] (13)

where \( U^0 \) stands for the work performed by the pre-buckling stresses, defined in (12a)-(12c), on the non-linear (quadratic) strain components. Then, (13) can be transformed into (Basaglia et al. 2013)

\[
U = \frac{1}{2} \int_L \left( \varphi_{,xx}^T C \varphi_{,xx} + \varphi_{,x}^T D \varphi_{,x} + \varphi_{,s}^T E \varphi_{,s} + \varphi_{,xx}^T E \varphi_{,xx} + \lambda \{ \varphi_{,xx}^T [\varphi_j^{0,s} - \varphi_j^{0,s}] + \varphi_{,x}^T [\varphi_j^{0,s} - \varphi_j^{0,s}] \} \right) dx = 0
\] (14)
where $1 \leq j \leq n_d^0$ and the four lines in the integrand include terms associated with, respectively, the (i) linear stiffness (the same already appearing in (4)) and (ii) geometric stiffness associated with the $\sigma_{xx}^0, \sigma_{ss}^0$ and $\tau_{xs}^0$ pre-buckling stress components. Moreover, $\lambda$ is the load parameter and $X_j^{C-x}, X_j^{C-xp}, X_j^{C-sp}, X_j^{C-s}$ and $X_j^T$ are the cross-section geometric stiffness matrices associated with (i) normal longitudinal ($X_j^{C-x}$ and $X_j^{C-xp}$), (ii) normal transverse ($X_j^{C-sp}$ and $X_j^{C-s}$) and (iii) shear ($X_j^T$) pre-buckling stresses\(^7\) – their analytical expressions are provided in Appendix A. Note that the $\varphi_j^0(x)$ functions (components of $\varphi(x)$), which define the pre-buckling state, are already known at this point.

The member buckling analysis, i.e., the procedure required to solve (13), leading to the determination of $\lambda$ and $\varphi(x)$, can be performed either (i) analytically, only for simply supported members under uniform internal force/moment diagrams (the strong form of (13) is a system of differential equilibrium equations with sinusoids as exact solutions for $\varphi_k(x)$) or (ii) or numerically. The numerical solution of (13), which can be performed for any combination of loading and support conditions, involves the longitudinal discretization of the member into GBT-based beam finite elements – the finite element degrees of freedom are the nodal values and derivatives of the $\varphi_k(x)$ functions, approximated by means of cubic Lagrange and/or Hermite polynomials. Thus, the member mid-surface is (i) longitudinally discretized into GBT-based finite elements and (ii) transversally discretized into “wall segments” (limited by consecutive nodes), in a manner similar to a shell finite element mesh.

3. Application to Beams with Cross-Sections Containing Closed Cells

The application of the above GBT formulation to members (beams) with cross-sections containing closed cells is now addressed and illustrated, by analyzing the buckling behavior of the following two beams: (i) a rectangular hollow section (RHS) cantilever beam acted by two tip point loads (Fig. 3(a)) and (ii) a closed-flange I-section (CI) simply-supported beam subjected to an uniformly distributed load (Fig. 3(b)). In both cases the loads are applied either at the beam shear center axis ($P_{SC}, P_{SC}$) or on the top ($P_{TopTip}, P_{TopTip}$) or bottom ($P_{Bot}, P_{Bot}$) of the cross-section – in the latter cases, the applied load always passes through the cross-section shear center. For each cross-section shape, buckling curves are obtained for a broad range of lengths $L$: $100 \leq L \leq 10.000\ mm$. For validation purposes, the GBT results, obtained with the GBTUL 2.0 code are compared with ANSYS SFEA results.

![Figure 3](image)

**Figure 3:** Illustrative examples: (a) RHS cantilever acted by two tip point loads and (b) CSI simply-supported beam acted by a uniformly distributed load – the loads may act at any of the three locations indicated.

3.1 RHS Cantilevers under a Tip Point Load

Figs. (a)-(b) show the geometry of the RHS cantilever and the corresponding GBT nodal discretization adopted – the fairly large number of intermediate nodes considered in the webs is intended to achieve an adequate approximation of the stress component variations (namely that of $\sigma_{ss}$) in the vicinity of the load points of application. Figure 5 displays the out-of-plane and in-plane configurations of the most

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Note that the $X^{COP}$ terms stem from Poisson effects.
relevant deformation modes (out of a total of $N_d = 60$) obtained with the cross-section discretization shown in Fig. 4(b) – these deformation modes are (i) Conventional (global 1-3, distortional 5 and local 6-11), (ii) Shear (cell shear flow torsion 4, global 22-24 and local 25-34) and (iii) Transverse Extension (isotropic 41, deviatoric 42 and distortional 43-44).

Figure 5: Out-of-plane and in-plane configurations of the most relevant RHS deformation modes.

Figure 6 presents the cantilever buckling curves ($P_{cr} - L$) for two tip point loads acting on three levels (Fig. 3(a)): (i) top of the cross-section (GBT – Top), (ii) shear center level (GBT – SC) and (iii) bottom of the cross-section (GBT – Bottom) – this figure also displays some results provided by ANSYS SFEA, concerning the loads applies at the top (ANSYS – Top) and bottom (ANSYS – Bottom) of the free
Figure 6: Buckling curves ($P_{cr} - L$) of the RHS cantilevers subjected to bottom, shear center and top loading.

![Graph showing buckling curves](image)

$L = 200\,mm$  
$L = 320\,mm$  
$L = 1200\,mm$

Bottom  
Shear Center  
Top

Figure 7: GBTUL buckling mode shapes of the $L = \{200, 320, 1200\,mm\}$ RHS cantilevers subjected to bottom, shear center and top loading.

![Mode shapes](image)

end cross-section. On the other hand, Fig. 7 depicts the GBTUL 2.0 critical buckling mode shapes of the beams with lengths $L = \{200, 320, 1200\,mm\}$ for top and bottom loading. Finally, Fig. 8 provides GBTUL 2.0 representations of the pre-buckling stresses $\sigma_{xx}^0$, $\sigma_{zz}^0$, $\tau_{xz}^0$ for the $L = 320\,mm$ beam under bottom loading. The observation of the results presented in Figs. 6 to 8 leads to the following conclusions:
(i) The comparison with the critical buckling loads of the cantilevers loaded at the shear center level shows that changing the loading level only alters the values of such buckling loads for \((i)\) \(L < 450\, mm\), for bottom loading, and \((ii)\) \(L < 700\, mm\), for top loading – for longer beams, the buckling loads are virtually independent from the applied load level. In the shorter beams, moving the applied load level downwards/upwards increases/decreases the \(P_{cr}\) value.

(ii) The ANSYS critical buckling loads available, concerning both top and bottom loading, are in excellent agreement with the GBT values, as can be confirmed by looking at Table 1 – indeed, the differences are always below 3%.

Table 1: Comparison between the GBTUL and ANSYS buckling loads \((P_{cr})\) of the RHS cantilevers.

<table>
<thead>
<tr>
<th></th>
<th>(L) (mm)</th>
<th>(120)</th>
<th>(160)</th>
<th>(200)</th>
<th>(320)</th>
<th>(1200)</th>
<th>(2000)</th>
<th>(5000)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(P_{cr}) (kN)</td>
<td>GBTUL</td>
<td>ANSYS</td>
<td>GBTUL</td>
<td>ANSYS</td>
<td>GBTUL</td>
<td>ANSYS</td>
<td>GBTUL</td>
</tr>
<tr>
<td>Top</td>
<td>(\delta) (GBT/SFEM)</td>
<td>0.53%</td>
<td>0.51%</td>
<td>0.51%</td>
<td>0.63%</td>
<td>0.69%</td>
<td>-0.22%</td>
<td>0.54%</td>
</tr>
<tr>
<td>(P_{cr}) (kN)</td>
<td>GBTUL</td>
<td>40.91</td>
<td>40.80</td>
<td>40.79</td>
<td>40.84</td>
<td>23.54</td>
<td>13.52</td>
<td>5.24</td>
</tr>
<tr>
<td>ANSYS</td>
<td>40.69</td>
<td>40.59</td>
<td>40.59</td>
<td>40.59</td>
<td>23.38</td>
<td>13.55</td>
<td>5.21</td>
<td></td>
</tr>
<tr>
<td>Bottom</td>
<td>(\delta) (GBT/SFEM)</td>
<td>-2.91%</td>
<td>-2.07%</td>
<td>-1.37%</td>
<td>-1.57%</td>
<td>0.69%</td>
<td>-0.22%</td>
<td>0.54%</td>
</tr>
<tr>
<td>(P_{cr}) (kN)</td>
<td>GBTUL</td>
<td>302.84</td>
<td>215.86</td>
<td>173.39</td>
<td>101.46</td>
<td>23.54</td>
<td>13.52</td>
<td>5.24</td>
</tr>
<tr>
<td>ANSYS</td>
<td>311.93</td>
<td>220.43</td>
<td>175.80</td>
<td>103.08</td>
<td>23.38</td>
<td>13.55</td>
<td>5.21</td>
<td></td>
</tr>
</tbody>
</table>

(iii) Figure 7 makes it possible to assess the critical buckling mode shape changes caused by moving the applied loads upwards and downwards from the shear center level. In latter case (bottom loading) and regardless of the length, the cantilever always buckles in a mode involving local deformation in the vicinity of the support and, naturally, in the region corresponding to the cross-section bottom-half – this critical buckling mode combines sizeable participations from the GBT deformation modes\(^8\) 8 (~40%), 10 (~30%), 6 (~20%) (see Fig. 5). Concerning the other cases (shear center and top loading), the shorter beams \((L < 700\, mm)\) exhibit different critical buckling mode shapes, namely \((iii)\) a local-distortional mode, combining contributions from deformation modes 7 (~38%), 5 (~35%), 9 (~15%), 12 (~8%) and involving mostly anti-symmetric deformations in the free end region (shear center loading), and \((iii)\) a localized mode, occurring at the web regions close to the free end (where the loads are applied) and combining contributions from deformation modes 6 (~73%), 8 (~22%), 11 (~3%) (top loading).

(iv) While for bottom loading the instability is caused by the compressive normal longitudinal stresses \(\sigma_{xx}\) acting on the beam bottom-half support region (where these stresses are higher), for the other cases such instability is triggered by the compressive normal transverse stresses \(\sigma_{ss}\) acting on the free end web regions located below the load application points. Note that these stresses are tensile in the case of bottom loading. These remarks provide the explanation for the differences between the initial portions (smaller lengths) three buckling curves depicted in Fig. 6. For the larger lengths, the normal longitudinal stresses acting in the close vicinity of the end support become dominant and, therefore, the position of the loads ceases to be relevant.

\(^8\) The percentage values provided after each GBT deformation mode provide its modal participation factor, which “measures” the relative importance of that deformation mode on the beams critical buckling mode configuration. In GBTUL 2.0, the modal participation factors are calculated as the ratio between the internal strain energies associated with (i) the contribution of the deformation mode under consideration and (ii) the whole critical buckling mode (Bebiano et al. 2015b).
The GBTUL pre-buckling stress distributions \( \sigma_{xx}^0, \sigma_{ss}^0, \tau_{xs}^0 \) displayed in Fig. 8 exhibit the expected features: (v1) \( \sigma_{xx}^0 \) vary linearly along the cross-section height and their magnitudes increase gradually as one travels from the free to the fixed end (the tension stress region is barely visible), (v2) \( \sigma_{ss}^0 \) consist essentially of two web “bulbs”, located in the regions under the two applied loads and (v3) the highest \( \tau_{xs}^0 \) also occur at those same regions (note that the shear force is uniform). It is still worth mentioning that the ANSYS stress distributions corresponding to those depicted in Fig. 8 are not shown here, because of the different color spectra adopted by the two codes (and the lack of time to harmonize them, so that a straightforward direct comparison becomes possible).

![Figure 8: GBTUL pre-buckling stress (\( \sigma_{xx}^0, \sigma_{ss}^0, \tau_{xs}^0 \)) distributions for the \( L = 320 \text{mm} \) RHS cantilever under bottom loading.](image)

3.2 Example 2: Closed-Flange I-Section Simply Supported Beams

Figures 9(a)-(b) show the geometry and nodal discretization adopted of the closed-flange I-section (CIS) considered in this work\(^9\). Figure 10 displays the configurations of the most important deformation modes (out of \( N_d = 60 \)) provided by the cross-section analysis – they are (i) Conventional (global 1-3, distortional 6-10 and local 11-15), (ii) Shear (cell shear flow 4-5\(^10\), global 26-34 and local 35-36) and (iii) Transverse Extension (isotropic 50, deviatoric 51-56 and distortional 57-58).

Figure 11 shows the buckling curves (\( p_{cr} - L \)) of the CIS simply supported beams subjected to a uniformly distributed load acting on three locations depicted in Fig. 3(b), namely (i) on the top web-flange intersection (GBT – Top), (ii) at the shear center (GBT – SC) and (iii) on the bottom web-flange intersection (GBT – Bottom). This figure also includes some ANSYS SFEA values concerning top (ANSYS – Top) and bottom (ANSYS – Bottom) loading. As for Fig. 12, it depicts the critical buckling mode shapes provided by the GBTUL analyses of the \( L = \{500, 1700, 4000\text{mm} \} \) beams subjected to the three loadings considered. The observation of the buckling results presented in Figs. 10 and 11 prompts the following remarks:

![Figure 9: Closed-flange I-section (CIS) (a) geometry and (b) GBT nodal discretization.](image)

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\(^9\) This cross-section geometry corresponds to two \( 200 \times 45 \times 15 \times 1.6 \) LiteSteel Beam cross-sections placed “back-to-back” with only one web and single-cell flanges (the internal walls are removed) – note that LiteSteel Beams with the above cross-section dimensions were recently tested by Wan & Mahendran (2015).

\(^10\) While mode 4 is associated with torsion, there is no particular structural meaning attributed to mode 5 (the second cell shear flow mode).
Figure 10: Out-of-plane and in-plane configurations of the most relevant CIS deformation modes.

(i) Like in the RHS cantilever analyzed previously, the buckling curves associated with bottom/top load lie significantly above/below the one associated with shear center loading for the shorter beams \((L < 600\, mm)\). For larger lengths, the three curves gradually merge together, even if they remain at least 5% apart within the length range considered in this work. Note that, for \(1800 \leq L \leq 3000\, mm\), the critical buckling loads associated with bottom and shear center loading are practically coincident.

(ii) Table 2 provides the means to compare the GBTUL and ANSYS \(p_{cr}\) values that are also displayed in Fig. 11. It is observed that for all but one beam the differences between the two sets of critical
Figure 11: Buckling curves ($p_{cr} - L$) of the CIS beams subjected to bottom, shear center and top loading.

$L = 500mm$  
$L = 1700mm$  
$L = 4000mm$

Figure 12: GBTUL critical buckling mode shapes of the $L = \{500, 1700, 4000mm\}$ CIS beams subjected to bottom, shear center and top loading.

buckling loads never exceeds 5%. The exception is the $L = 200mm$ beam under bottom loading, for which the difference is of about 10%, due to the fact that the GBT discretization adopted was not fine enough.\(^{11}\)

\(^{11}\) The nodal discretization adopted did not enable capturing localized deformations occurring at the bottom part of the web – considering 7 (instead of 5) web intermediate nodes leads to $p_{cr} = 2.568kN/mm$ ($\delta = -1.37\%$).
For $L < 1700\text{mm}$, all the beams buckle in modes that exhibit predominantly shear deformations in the end regions (near the supports) and combine visible participations from the GBT deformation modes 11 ($\sim 70\%$), 12 ($\sim 23\%$), 9 ($\sim 5\%$) (see Fig. 10). This critical buckling mode is also shared by longer beams ($1800 \leq L \leq 3000\text{mm}$) under top loading. However, the same beams, when subjected to bottom or shear center loading, buckle in local modes triggered by the compressive normal longitudinal stresses acting on the central region of the top flange upper wall. These modes consist almost exclusively of GBT deformation mode 14 ($\sim 95\%$) and, since the above compressive stresses are independent of the loading point of application, the buckling curves associated with bottom and shear center loading coincide for this length range. Finally, the longer beam buckle in a global lateral-torsional mode, exhibiting similar contributions from modes 3 and 4 and a more or less relevant presence of the distortional mode 6. Although the buckling mode is identical, and as it is already well known, moving the applied load level downwards/upwards increases/decreases the $p_{cr}$ value – for this particular case, the differences are in the range $5\% < |\delta| < 18\%$.

4. Conclusion

The GBT formulation developed by Basaglia & Camotim (2013), which makes it possible to analyze the buckling behavior of thin-walled members acted by arbitrary loadings (namely including transverse loads applied away from the shear center axis), was briefly reviewed and employed to analyze beams with two cross-section geometries involving closed cells – the formulation had never before been employed to analyze members with such cross-sections. The beams analyzed consisted of (i) rectangular hollow section (RHS) cantilevers acted by two tip point loads and (ii) closed-flange I-section (CIS) simply-supported beams acted by a uniformly distributed load – in both cases, the loadings were applied above, at and below the shear center axis.

In order to validate the application of the GBT formulation to the beams described in the previous paragraph, the GBT-based buckling results were compared with values yielded by ANSYS shell finite element analyses. An excellent correlation was obtained in all cases, regardless of the buckling mode nature (local, distortional, localized, shear, flexural-torsional), thus showing that the aforementioned formulation provides accurate buckling results in the context of beams with cross-sections containing closed cells. Moreover, it was shown that, as either know or expected, the distance between the point of load application and the shear center axis may influence significantly the beam (i) buckling load and/or (ii) buckling mode shapes, particularly for the shorter lengths. This stems mostly from the development of

Table 2: Comparison between the GBTUL and ANSYS buckling loads ($p_{cr}$) of the CIS simply supported beams.

<table>
<thead>
<tr>
<th>$L\text{ (mm)}$</th>
<th>200</th>
<th>500</th>
<th>1000</th>
<th>2000</th>
<th>4000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Top $p_{cr}$ (kN/mm) GBTUL</td>
<td>0.248</td>
<td>0.152</td>
<td>0.092</td>
<td>0.043</td>
<td>0.009</td>
</tr>
<tr>
<td>ANSYS</td>
<td>0.260</td>
<td>0.154</td>
<td>0.095</td>
<td>0.043</td>
<td>0.009</td>
</tr>
<tr>
<td>$\delta$ (GBT/SFEM)</td>
<td>-4.63%</td>
<td>-1.55%</td>
<td>-2.35%</td>
<td>-1.82%</td>
<td>-0.13%</td>
</tr>
<tr>
<td>Bottom $p_{cr}$ (kN/mm) GBTUL</td>
<td>2.458</td>
<td>0.442</td>
<td>0.142</td>
<td>0.048</td>
<td>0.013</td>
</tr>
<tr>
<td>ANSYS</td>
<td>2.239</td>
<td>0.445</td>
<td>0.148</td>
<td>0.048</td>
<td>0.012</td>
</tr>
<tr>
<td>$\delta$ (GBT/SFEM)</td>
<td>9.76%</td>
<td>-0.59%</td>
<td>-3.89%</td>
<td>0.26%</td>
<td>4.48%</td>
</tr>
</tbody>
</table>
quite large compressive normal transverse stresses in the vicinity of the load point of application, which may be responsible for triggering different buckling phenomena (e.g., localized instability).

Final, one last word to mention that none of the problems considered in this work involve pre-buckling shear stresses associated with cell shear flow modes 4 and 5 (see Fig. 13(b) and note that mode 4 stands for torsion). In order to check whether the available GBT formulation also provides accurate buckling results in this special case, it is necessary to analyze members with cross-section shapes and loadings such as those depicted in Figs. 13(a), concerning a two-cell RHS member subjected to transverse loads acting either on one external web or on the central web. In the former case, which involves torsion, part of the pre-buckling shear stresses are associated with both modes 4 and 5, whose shear flows are displayed in Fig. 13(c). In the latter case, not involving torsion, only shear stresses associated with mode 5 appear in the pre-buckling stage. The authors are currently investigating the application of the GBT formulation to this type of problems – the output of this research effort will be reported in the near future.

![Diagram](image)

Figure 13: Two-cell RHS (a) loadings causing pre-buckling cell shear flow deformation associated with (a1) torsion and (a2) distortion, (b1)-(b2) in-plane shapes and (c1)-(c2) cell shear flows of the GBT deformation modes 4-5.

Acknowledgments

The first author gratefully acknowledges the financial support of FCT (Fundaçao para a Ciência e Tecnologia – Portugal), through the post-doctoral scholarship n° SFRH/BPD/98111/2013.

References


Appendix A

The components of the GBT cross-section stiffness matrices read

\[ C_{ik} = \int_{S} \frac{Et}{1-v^2} u_i u_k ds + \int_{S} \frac{Et^3}{12(1-v^2)} w_i w_k ds \]  \hspace{1cm} (A.1)

\[ B_{ik} = \int_{S} \frac{Et}{1-v^2} v_i v_k ds + \int_{S} \frac{Et^3}{12(1-v^2)} w_i v_{k,s} ds \]  \hspace{1cm} (A.2)

\[ D_{ik} = \int_{S} Gt(u_i v_k + v_i u_k) ds + \int_{S} \frac{Et^3}{3} w_i v_k ds \]  \hspace{1cm} (A.3)

\[ E_{ik} = \int_{S} \frac{vEt}{1-v^2} u_i v_k ds + \int_{S} \frac{vEt^3}{12(1-v^2)} w_i v_{k,s} ds \]  \hspace{1cm} (A.4)

\[ X_{jik}^{\sigma-x} = \int_{S} \frac{Et}{1-v^2} u_j (v_i v_k + w_i w_k) ds \]  \hspace{1cm} (A.5)

\[ X_{jik}^{\sigma-xp} = \int_{S} \frac{vEt}{1-v^2} v_j (v_i v_k + w_i w_k) ds \]  \hspace{1cm} (A.6)

\[ X_{jik}^{\sigma-xp} = \int_{S} \frac{vEt}{1-v^2} u_j w_{i,s} v_{k,s} ds \]  \hspace{1cm} (A.7)

\[ X_{jik}^{\sigma-xp} = \int_{S} \frac{Et}{1-v^2} v_j w_{i,s} w_{k,s} ds \]  \hspace{1cm} (A.8)

\[ X_{jik}^{x} = \int_{S} Gt(u_{j,s} + v_j) (v_{i,s} v_k + w_{i,s} w_k) ds \]  \hspace{1cm} (A.9)

where (i) \( E, G, \nu \) are the material Young’s modulus, shear modulus and Poisson’s ratio, (ii) subscripts \( i, j, k \) span the deformation mode set \( (1,...,N_d) \) and (iii) \( S \) stands for the cross-section mid-line domain. The first and second terms of (A.1)-(A.6) concern membrane and bending properties, respectively.