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Dynamic Stability of Planar Frames Supported by Elastic Foundation

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Abstract
An exact analytical solution for a vibrating beam-column element on an elastic Winkler foundation is derived. The solution covers all cases comprised of constant compressive and tensile axial force with restrictions of $k_s - m\omega^2 > 0$ and $k_s - m\omega^2 < 0$. Closed form solutions of dynamic shape functions are explicitly derived for each case and they are used to obtain frequency-dependent dynamic stiffness terms. Governing dynamic equilibrium equations are not only enforced at element ends, but also at any point along the element. To this end, derived stiffness terms are exact and they include distributed mass effects and geometric nonlinear effects such as axial-bending coupling. For this reason, the proposed solution eliminates the need of further element discretization to obtain more accurate results. In absence of elastic foundation (i.e., $k_s \to 0$), exact dynamic stiffness terms for beam-columns are also derived and presented in this study. Derived stiffness terms are implemented in a software program and several examples are provided to demonstrate the potential of the present study.

1. Introduction

Analysis of beams supported by elastic or visco-elastic media is very common in engineering practice. Beams on elastic foundation can be subjected to transverse loads as well as axial loads. This is commonly encountered in many diverse problems such as end bearing piles, buried pipelines, reinforcing filaments in composite materials, frames resting on or buried in soil.

Application of the Winkler foundation model dates back to 1867. Due to its simplicity, the Winkler model is well suited for many applications and it has been a very popular area of interest for many researchers (Kerr 1964, Scott 1981, Eisenberger and Yankelevsky 1985, Yankelevsky and Eisenberger 1986, Williams and Kennedy 1987). Pasternak (1954) and Kerr (1964) accounted for interaction between foundation springs, enabling a wealth of further applications (Zhaohua and Cook 1983, Williams and Kennedy 1987, Razaqpur and Shah 1991).

In early finite element applications, formulations based on cubic Hermitian functions were used to derive stiffness terms for beams on elastic foundation. These solutions are approximate

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because the assumed shape functions only resemble the displacement field and hence, it is necessary to use several elements per member to achieve an acceptable accuracy in analysis results (Cook et al. 2001). Exact (static) stiffness matrices were derived in other studies (e.g., Eisenberger, Yankelevsky 1985, 1986, Williams and Kennedy 1987 and Alemdar, Gülkan 1997). In these studies, governing equilibrium equations are expressed in differential equation forms and they are solved to obtain exact shape functions that are also used to derive exact stiffness terms. As a sequel, Gülkan and Alemdar (1999) derived exact shape functions and stiffness matrices for beams on two-parameter elastic foundation.

Vibration, stability and dynamic response of axially loaded beams on elastic foundation were further studied by several researches (Kim 2004, 2005, Spyrakos and Beskos 1982, and Arboleda-Monsalve, et. al. 2008). The interaction between structural components and the adjacent bonded media is of fundamental importance not only for foundation design but also as a classical problem for applied mechanics, so it has attracted the interest of researchers and engineers. Wang et al. (2005) reviewed the state-of-the-art in this field, highlighting the key areas of development, including the modeling of the soil media and various analytical as well as numerical approaches in analyzing the interaction between the foundation and soil. Shufrin and Eisenberger (2006a and 2006b) investigated effect of stability and vibration of shear -deformable plates. Peiris et al. researched the soil-pile interaction of a pile embedded in a deep multi-layered soil under seismic excitation considering both kinematic and inertial interaction effects.

This study presents a finite element solution for vibrating beam-column on elastic foundation, subjected to a constant axial load. Exact dynamic shape functions are derived in order to obtain frequency-dependent dynamic stiffness terms. The solution domain includes four different cases, depending on constant compressive or tensile axial force within the ranges of $k_s - m\omega^2 > 0$ and $k_s - m\omega^2 < 0$. Transition is ensured for the case when $k_s - m\omega^2 = 0$. Each case is studied separately. Geometric nonlinear effects (i.e., axial-bending coupling) and distributed mass effects are directly included within the derived stiffness terms. In Sections 2 and 3, the governing dynamic equilibrium equation is expressed in a differential equation form and the equilibrium equation is solved to obtain shape functions after enforcing essential boundary conditions. These shape functions form the basis for obtaining dynamic stiffness terms, which is demonstrated in Section 4. These results are used to derive solutions of other engineering problems, such as dynamic/static stability analysis of beams without elastic foundation. This is further elaborated in Section 5. Numerical examples are provided in the following section to demonstrate the merits of the proposed solution. It should be noted that derived frequency dependent shape functions and dynamic stiffness terms are exact and distributed mass and geometric nonlinear effects (axial-bending coupling) are directly included in these terms. This ensures that the proposed solution strictly satisfies equilibrium equations, not only at the element ends but also within the element. For this reason, one element per member suffice to obtain exact solutions whereas such accuracy can be only achieved with using more than one element if cubic Hermitian type beam elements are employed.

2. Derivation of Governing Differential Equilibrium Equation

The beam element consists of three degrees of freedom at each end: one horizontal, one vertical and one rotational. The element formulation adopts an Euler-Bernoulli type beam element formulation. Forces on an infinitesimal segment on the element are shown in Fig. 1.

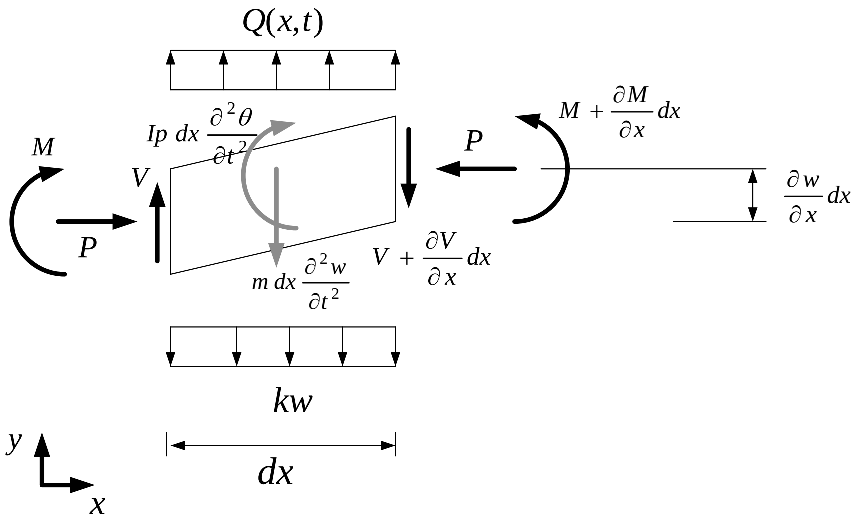


Figure 1: Dynamic Equilibrium Forces Shown on Infinitesimal Euler-Bernoulli Type Beam Element

Referring to Fig. 1, the governing equilibrium equation is obtained as

$$EI\frac{\partial^4 w(x,t)}{\partial x^4}+P\frac{\partial^2 w(x,t)}{\partial x^2}+k_s w(x,t)+m\frac{\partial^2 w(x,t)}{\partial t^2}=Q(x,t) \tag{1}$$

in which EI is flexural rigidity of the element, P is the constant (compressive) axial load, k_s is the Winkler foundation parameter, m is mass per unit length, $Q(x,t)$ is transverse distributed load and $w(x,t)$ is the transverse displacement along the element. In the following development, the load $Q(x,t)$ is not considered. In addition, it is assumed that rotational inertia effects are negligible (i.e., $I_p\partial^2\theta/\partial t^2 \approx 0$) on account of slenderness.

The partial differential equation given in Eq. (1) can be solved by separation of variables method. The displacement $w(x,t)$ is now expressed as

$$w(x,t)=y(x)g(t) \tag{2}$$

Then,

$$\frac{1}{m\,y(x)}\left(EI\frac{d^4y(x)}{dx^4}+P\frac{d^2y(x)}{dx^2}+k_s\,y(x)\right)=-\frac{1}{g(t)}\frac{d^2g(t)}{dt^2} \tag{3}$$

A nontrivial solution is possible when Eq. (3) equals a constant, ω^2 :

$$EI\frac{d^4y(x)}{dx^4}+P\frac{d^2y(x)}{dx^2}-\left(m\omega^2-k_s\right)y(x)=0 \tag{4}$$

$$\frac{d^2g(t)}{dt^2}+\omega^2\,g(t)=0 \tag{5}$$

The term ω is the circular frequency of the governing equation. Note that Eq. (1) is expressed in the time-domain whereas Eq. (4) is presented in the frequency-domain. Finally, Eq. (4) is further expressed in the following form:

$$\frac{d^4 y(x)}{dx^4} + A\frac{d^2 y(x)}{dx^2} - By(x) = 0 \tag{6}$$

where

$$A = \frac{P}{EI} \qquad B = \frac{m\omega^2 - k_s}{EI} \tag{7}$$

The solution for Eq. (5) is

$$g(t) = g_1 Sin[\omega t] + g_2 Cos[\omega t] \tag{8}$$

where g_1 and g_2 are constants that can be found from the initial prescribed displacement and velocity patterns. In the following sections, the element formulation is focused only on Eq. (6).

3. Solution of Governing Equilibrium Equations

Based on having tensile or compressive axial load in the beam as well as having negative or positive values of $(m\omega^2 - k_s)$, other possible cases can be derived from Eq. (6). Therefore, a total of four different cases is identified. They are studied separately in this section. It is noted that many possible combinations of ω^2 and k_s can yield the same value for $(m\omega^2 - k_s)$ so the solution space implied by the indicated ranges is very wide. Due to limited space, only a partial set of results is given in the present paper. In all cases presented in this section, exact forms of shape (interpolation) functions are obtained. Then, these shape functions are used to get exact dynamic stability stiffness terms, which are covered in the next section.

2.1 Case 1: Beam-column subjected to a constant compressive axial force and $m\omega^2 - k_s > 0$

$$\frac{d^4 y}{dx^4} + A\frac{d^2 y}{dx^2} - By = 0 \tag{9}$$

and

$$A = \frac{|P|}{EI} \qquad B = \frac{|m\omega^2 - k_s|}{EI} \tag{10}$$

The roots of the characteristic equation of Eq. (9) are

$$D_1 = -\beta i \qquad D_2 = \beta i \qquad D_3 = -\gamma \qquad D_4 = \gamma \tag{11}$$

where $i = \sqrt{-1}$ and

$$\beta = \frac{\sqrt{A + \sqrt{A^2 + 4B}}}{\sqrt{2}} \qquad \gamma = \frac{\sqrt{-A + \sqrt{A^2 + 4B}}}{\sqrt{2}} \tag{12}$$

provided that $\sqrt{A^2 + 4B} > 0$. Then, the complementary solution for Eq. (9) is

$$y(x) = c_1 Cos[\beta x] + c_2 Cosh[\gamma x] + c_3 Sin[\beta x] + c_4 Sinh[\gamma x] \tag{13}$$

The constants $c_1 - c_4$ can be obtained after substituting the following essential boundary conditions:

$$y(0) = v_1 \qquad \frac{d\,y(0)}{d\,x} = \theta_1 \qquad y(L) = v_2 \qquad \frac{d\,y(L)}{d\,x} = \theta_2 \tag{14}$$

Then,

$$\begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \mathbf{H} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix} \tag{15}$$

The matrix $\mathbf{H}$ is constructed by substituting the boundary conditions into Eq. (13). Finally, Eq. (15) is solved for $c_1 - c_4$ and then, this result is substituted into Eq. (13) to obtain the following:

$$y(x) = \lfloor N_1 \quad N_2 \quad N_3 \quad N_4 \rfloor \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{16}$$

The functions $N_1 - N_4$ are referred to as exact shape functions because they are directly derived from the solution of Eq. (9). It is verified that $N_1 - N_4$ converge to the cubic Hermitian polynomials at the limit of $k_s \to 0$, $P \to 0$ and $\omega \to 0$. Closed form solution of the shape functions for Case 1 is given in the Appendix A.

2.2 Case 2: Beam-column subjected to a constant tensile axial force and $m\omega^2 - k_s > 0$

$$\frac{d^4 y}{d x^4} - A \frac{d^2 y}{d x^2} - B y = 0 \tag{17}$$

in which the terms A and B are defined in Eq. (10). Note that $\sqrt{A^2 + 4B} > 0$. One can employ the following characteristics roots:

$$D_1 = \beta \qquad D_2 = -\beta \qquad D_3 = \gamma \qquad D_4 = -\gamma \tag{18}$$

where

$$\beta = \frac{\sqrt{-A + \sqrt{A^2 + 4B}}}{\sqrt{2}} \qquad \gamma = \frac{\sqrt{A + \sqrt{A^2 + 4B}}}{\sqrt{2}} \tag{19}$$

Then, the complementary solution for Eq. (17) takes the same form as given in Eq. (13).

2.3 Case 3: Beam-column subjected to a constant compressive axial force and $m\omega^2 - k_s < 0$

$$\frac{d^4 y}{d x^4} + A \frac{d^2 y}{d x^2} + B y = 0 \tag{20}$$

in which the terms A and B are defined in Eq. (10). There are three possible sets of solutions: the cases for $A < 2\sqrt{B}$, $A = 2\sqrt{B}$ and $A > 2\sqrt{B}$. In the following, each sub-case is studied separately. In each case, the complementary solution of the corresponding equation is given explicitly to derive shape functions similar to the method explained for Case 1.

$A < 2\sqrt{B}$: The roots of the characteristic equation given in Eq. (20) are

$$D_1 = \beta \qquad D_2 = -\beta \qquad D_3 = \gamma \qquad D_4 = -\gamma \tag{21}$$

where

$$\beta = \frac{\sqrt{-A + i\sqrt{4B - A^2}}}{\sqrt{2}} \qquad \gamma = \frac{\sqrt{-A - i\sqrt{4B - A^2}}}{\sqrt{2}} \tag{22}$$

Then, the complementary solution for Eq. (20) becomes

$$\begin{aligned} y(x) = {} & c_1\, Cosh[\eta\, x]\, Cos[\xi\, x] + c_2\, Cosh[\eta\, x]\, Sin[\xi\, x] + \\ & c_3\, Sinh[\eta\, x]\, Cos[\xi\, x] + c_4\, Sinh[\eta\, x]\, Sin[\xi\, x] \end{aligned} \tag{23}$$

in which

$$\xi = \sqrt{\lambda^2 + \delta} \qquad \eta = \sqrt{\lambda^2 - \delta} \tag{24}$$

And

$$\delta = \frac{A}{4} \qquad \lambda = \sqrt[4]{\frac{B}{4}} \tag{25}$$

$A = 2\sqrt{B}$: The roots of the characteristic equation given in Eq. (20) are

$$D_1 = \beta\, i \qquad D_2 = -\beta\, i \qquad D_3 = \gamma\, i \qquad D_4 = -\gamma\, i \tag{26}$$

And

$$\beta = \gamma = \frac{\sqrt{A}}{\sqrt{2}} \tag{27}$$

Then the corresponding complementary solution takes the following form:

$$y(x) = c_1\, Cos[\beta\, x] + c_2\, x\, Cos[\beta\, x] + c_3\, Sin[\beta\, x] + c_4\, x\, Sin[\beta\, x] \tag{28}$$

$A > 2\sqrt{B}$: The roots of the characteristic equation given in Eq. (20) are

$$D_1 = \beta\, i \qquad D_2 = -\beta\, i \qquad D_3 = \gamma\, i \qquad D_4 = -\gamma\, i \tag{29}$$

where

$$\beta = \frac{\sqrt{A - \sqrt{A^2 - 4B}}}{\sqrt{2}} \qquad \gamma = \frac{\sqrt{A + \sqrt{A^2 - 4B}}}{\sqrt{2}} \tag{30}$$

Then the corresponding complementary solution takes the following form:

$$y(x) = c_1\, Cos[\beta\, x] + c_2\, Cos[\gamma\, x] + c_3\, Sin[\beta\, x] + c_4\, Sin[\gamma\, x] \tag{31}$$

2.4 Case 4: Beam-column subjected to a constant tensile axial force and $m\omega^2 - k_s < 0$

$$\frac{d^4 y}{dx^4} - A\frac{d^2 y}{dx^2} + By = 0 \tag{32}$$

where the terms A and B are defined in Eq. (10). There are three possible sets of solutions: the cases for $A < 2\sqrt{B}$, $A = 2\sqrt{B}$ and $A > 2\sqrt{B}$. In the following, each sub-case is studied separately.

$A < 2\sqrt{B}$: The roots of the characteristic equation given in Eq. (32) are

$$D_1 = \beta \qquad D_2 = -\beta \qquad D_3 = \gamma \qquad D_4 = -\gamma \tag{33}$$

where

$$\beta = \frac{\sqrt{A + i\sqrt{4B - A^2}}}{\sqrt{2}} \qquad \gamma = \frac{\sqrt{A - i\sqrt{4B - A^2}}}{\sqrt{2}} \tag{34}$$

The corresponding complementary solution takes the following form:

$$\begin{gathered} y(x) = c_1\, Cosh[\xi\, x]\, Cos[\eta\, x] + c_2\, Cosh[\xi\, x]\, Sin[\eta\, x] + \\ c_3\, Sinh[\xi\, x]\, Cos[\eta\, x] + c_4\, Sinh[\xi\, x]\, Sin[\eta\, x] \end{gathered} \tag{35}$$

in which the terms ξ , η , δ and λ are defined in Eq.(24) and Eq. (25).

$A = 2\sqrt{B}$: The roots of the characteristic equation given in Eq. (20) are

$$D_1 = \beta \qquad D_2 = -\beta \qquad D_3 = \gamma \qquad D_4 = -\gamma \tag{36}$$

in which the terms β and γ are defined in Eq.(27). Then the corresponding complementary solution takes the following form:

$$y(x) = c_1\, Cosh[\beta\, x] + c_2\, x Cosh[\beta\, x] + c_3\, Sinh[\beta\, x] + c_4\, x Sinh[\beta\, x] \tag{37}$$

$A > 2\sqrt{B}$: The roots of the characteristic equation given in Eq. (20) are

$$D_1 = \beta \qquad D_2 = -\beta \qquad D_3 = \gamma \qquad D_4 = -\gamma \tag{38}$$

where

$$\beta = \frac{\sqrt{A + \sqrt{A^2 - 4B}}}{\sqrt{2}} \qquad \gamma = \frac{\sqrt{A - \sqrt{A^2 - 4B}}}{\sqrt{2}} \tag{39}$$

Then the corresponding complementary solution takes the following form:

$$y(x) = c_1\, Cosh[\beta\, x] + c_2\, Cosh[\gamma\, x] + c_3\, Sinh[\beta\, x] + c_4\, Sinh[\gamma\, x] \tag{40}$$

4. Dynamic Stiffness Terms

Dynamic stiffness terms for Case 1 are obtained from the following equation:

$$K_{ij} = EI\int_0^L \frac{d^2 N_i}{dx^2}\frac{d^2 N_j}{dx^2}\, dx - P\int_0^L \frac{dN_i}{dx}\frac{dN_j}{dx}\, dx - \left(m\,\omega^2 - k_s\right)\int_0^L N_i N_j\, dx \tag{41}$$

in which the first integral gives material stiffness terms, the second integral is for the element geometric stiffness terms and the third integral is for the stiffness terms attributed to Winkler foundation and dynamic effects. The term N_i represents i^{th} shape function, which is obtained in the previous section separately for each case. All of the integrals run over the element length L. It should be noted from Eq. (41) that the second and the third integrals have a destabilizing effect. Dynamic stiffness terms for other cases can be similarly expressed as follows:

Case 2:

$$K_{ij} = EI\int_0^L \frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}dx + P\int_0^L \frac{dN_i}{dx}\frac{dN_j}{dx}dx - \left(m\omega^2 - k_s\right)\int_0^L N_iN_jdx \tag{42}$$

Case 3:

$$K_{ij} = EI\int_0^L \frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}dx - P\int_0^L \frac{dN_i}{dx}\frac{dN_j}{dx}dx + \left(m\omega^2 - k_s\right)\int_0^L N_iN_jdx \tag{43}$$

Case 4:

$$K_{ij} = EI\int_0^L \frac{d^2N_i}{dx^2}\frac{d^2N_j}{dx^2}dx + P\int_0^L \frac{dN_i}{dx}\frac{dN_j}{dx}dx + \left(m\omega^2 - k_s\right)\int_0^L N_iN_jdx \tag{44}$$

5. Selected Engineering Problem Types

Sections 3 and 4 covers all possible cases for beams supported by an elastic foundation and subjected to a compressive or tensile axial load. The stiffness terms derived are expressed in terms of P (axial load), ω (circular frequency), m (mass per unit length), k_s (elastic foundation parameter), EI (flexural rigidity) and L (element length). It is also possible to generate solutions for other types of engineering problems from these results. For instance, the derived dynamic stiffness terms reduces to static response of a beam element in the absence of elastic foundation and vibration frequency (i.e., $k_s \to 0$ and $\omega \to 0$). This can be achieved either by substituting very small values for k_s and ω or by taking a limit of stiffness terms while k_s and ω approaching zero.

Table 1 summarizes problem types that are selected in this study. Closed forms of stiffness terms for these problems are explicitly derived and given in the Appendices.

Table 1: List of Engineering Problems Selected

Problem Definition	K	Case	Appendix
Dynamic Stability of Beams on Elastic Foundation	P, ω, m, k_s	1, 3	B,C
Dynamic Stability of Beams	P, ω, m	1	B[1] ($k_s = 0$)
Static Stability of Beams on Elastic Foundation	P, k_s	1	B[2] ($\omega = 0$)
Static Stability of Beams	P	1	D
Dynamic of Beams	ω, m	-	E

1. Use results given in Appendix B with $k_s \to 0$

2. Use results given in Appendix B with $\omega \to 0$

6. Numerical Examples

The stiffness terms derived in this study are added to a finite element library implemented in Mathematica (Wolfram 2015). Standard finite element procedures are followed for the examples provided in this section. This means that element stiffness matrices are first calculated and then, they are assembled into global stiffness matrix. The following *static-like* system of equations are repeatedly solved for a sequence of values of ω:

$$K(\omega)\Delta(\omega)=F(\omega) \tag{45}$$

where the vector $F(\omega)$ is assembled load vector expressed in frequency domain. It is demonstrated in this section that the proposed solution can be used in a typical finite element analysis framework such that more complex models can be addressed without any difficulty.

6.1. Vibration and Buckling Analysis of a Simply Supported Beam on Elastic Foundation

A simply supported beam resting on an elastic foundation is subjected to an axial compressive load. In order to find vibration frequencies of the beam, the determinate of stiffness matrix is first derived and then, the beam frequency (ω) is calculated in such a way that it is the frequency that makes the determinate vanishes. The determinate for this problem is given as follows:

$$g_1(P,\omega,m,k_s)=\frac{(EI)^2\left(\beta^2+\gamma^2\right)^2 Sin[\,L\,\beta\,]\,Sinh[L\,\gamma]}{-2\,\beta\,\gamma+2\,\beta\,\gamma\,Cos[\,L\,\beta\,]\,Cosh[L\,\gamma]+\left(\beta^2-\gamma^2\right)Sin[\,L\,\beta\,]\,Sinh[L\,\gamma]} \tag{46}$$

in which β and γ are defined in Eq. (12). Note that the above equation is derived from the stiffness terms obtained for Case 1. It is not possible to solve Eq. (46) for ω directly. Instead, a numerical solution is needed to find roots of Eq. (46).

In this example, axial compressive load level and foundation stiffness are varied and effect of these changes on the beam's fundamental frequency (ω_1) are investigated. The following numerical values are used: beam length L = 4.0 m. (157.5 in.); rectangular cross-section with width b = 40 mm (1.575 in) and depth d = 80 mm (3.1496 in); mass density ρ = 7850 kg/m^3; modulus of elasticity E = 2.1 x 10^{11} N/m^2 (30458 ksi). The elastic foundation parameter k_s is varied in such a way that $k_s/EI = 0,\ 0.05,\ 0.5,\ \text{and}\, 2.0$.

The relationship between axial compressive load and beam fundamental (first-vibration) frequency (ω_1) under different foundation stiffness is shown in Figure 2. These results are obtained by repeatedly solving Eq. (46) for different values of axial load and foundation stiffness. The curves represent P/P_e as abscissa and normalized foundation frequency ω_1/ω_{1o} as ordinate, in which $\omega 1_o = 73.69$ rad/s obtained for a vibrating beam in absence of both compressive load and foundation, and $P_e = \pi^2\, EI/L^2$. It is observed that the effect of foundation stiffness is negligible for $k_s/EI < 0.05$. For the values larger than this limit, the fundamental frequency increases as the foundation becomes stiffer. Similarly, the buckling load increases with increase of foundation stiffness. Table 2 tabulates numerical values for fundamental frequency of the beam at $P/Pe = 0$.

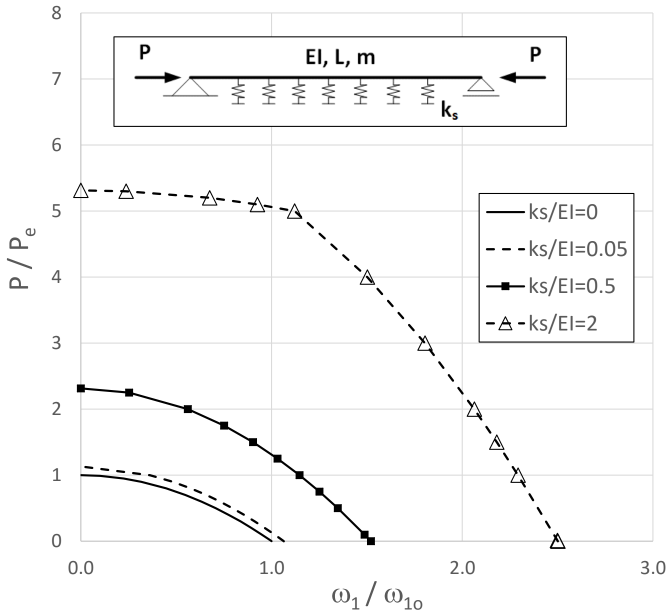


Figure 2: Variation of fundamental frequency of simply supported beam under different compressive axial load and foundation stiffness

Table 2: Fundamental Frequency of Simply Supported Beam at $P/Pe = 0$

k_s / EI	ω_1 / ω_{1o}
0	1
0.05	1.06
0.5	1.52
2	2.50

Buckling load of the beam is obtained by making the determinate of the stiffness matrix vanishes as ω approaches to zero. The following equation is derived under this condition and note that the buckling load is the axial load P that makes $g_2 = 0$:

$$g_2(P,k_s) = \frac{(EI)^2 (D+F)^2 Sin\left[\sqrt{\frac{F}{2}}\,L\right] Sinh\left[\sqrt{\frac{D}{2}}\,L\right]}{-4\sqrt{D}\left(-1+Cos\left[\sqrt{\frac{F}{2}}\,L\right]\ Cosh\left[\sqrt{\frac{D}{2}}\,L\right]+2\,(D-F)\,Sin\left[\sqrt{\frac{F}{2}}\,L\right]Sinh\left[\sqrt{\frac{D}{2}}\,L\right]\right)} \quad (47)$$

in which

$$A = \frac{P}{EI}, \quad B = -\frac{k_s}{EI}, \quad C = \sqrt{A^2 + 4\,B}, \quad D = -A + C, \quad F = A + C \quad (48)$$

A numerical solution is again needed to find buckling load from Eq. (47). This is exercised for the values k_s / EI selected and the results are given in Table 3.

Table 3: Buckling Load of Simply Supported Beam under Different Foundation Stiffness

k_s / EI	P / Pe
0	1
0.05	1.13
0.5	2.31
2	5.31

6.2. Moment Frame on Elastic Foundation

A steel moment frame supported by concrete columns and a concrete beam is studied in this example. The problem details are given in Fig. 3 (Arboleda-Monsalve et al., 2008). Both concrete columns and the concrete beam are underlain by an elastic foundation with k_s = 2.0684 N/mm^2 (0.3 kip/in^2). Steel members BC, CF and EF have sections of W14x26 with the following properties: E_s= 206,842.72 MPa (30,000 ksi), A= 4961 mm^2 (7.69 in^2), I=101.98 x 10^6 mm^4 (245 in^4) and m_s = 38.86 kg/m (56.32 x 10^{-7} kip-s^2/in^2). The concrete column members (AB and DE) have a diameter of 1000 mm (39.37 in) and m_{col} = 1886.88 kg/m (2734.34 x 10^{-7} kip-s^2/in^2). The concrete beam (BE) has a square section of 500mm x 500mm (19.68in x 19.68in) with m_{beam} = 600.0 kg/m (870.0 x 10^{-7} kip-s^2/in^2). Modulus of elasticity for concrete members is E_c = 25,998.75 MPa (3770.8 ksi). All members are assumed to be rigidly connected and the frame is fixed at points A and D. Axial deformations in all members are ignored.

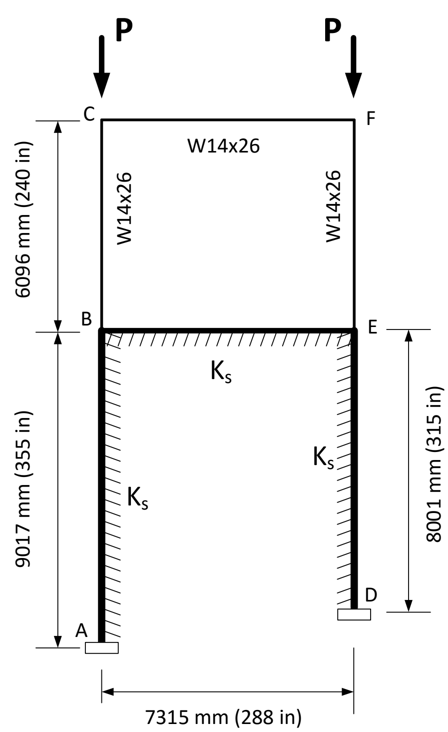


Figure 3: Moment Frame on Elastic Foundation

The present study predicts the buckling load (P_{cr}) as 3969.4 kN (892.35 kips). Arboleda-Monsalve et al. (2008) report the buckling load as 3762.6 kN (845.87 kips) in which shear deformations in the members and in the foundation are accounted for whereas such effects are ignored in this study. In the absence of axial load on columns, the current study also predicts the first fundamental vibration frequency (ω_1) as 42.33 rad/sec (6.74 Hz).

Figure 4 shows the plot of first-vibration frequencies calculated for different level of compressive axial loads. They are normalized by ω_1 and P_{cr}. It is noted that the solution for concrete columns supported on the elastic foundation switches from Case 1 (i.e., Eq. (9)) to Case 3 (i.e., Eq. (20) and $A < 2\sqrt{B}$) when $P/P_{cr} = 0.56$ (i.e., $P = 2224$ kN (500 kips)) whereas the solution for the concrete beam is Case 1 for all values of the axial load.

A similar (comparison) model is constructed with (Hermitian) beam finite elements (STAAD(X), 2015). In this case, each beam and column is modeled with 8 equal-length elements. The elastic foundation is represented with lumped springs placed at nodes. Member mass is uniformly distributed and lumped at element joints. With this model, buckling load is predicted as 3910.9 kN (879.2 kips) and the first fundamental frequency (in the absence of axial load) is found as 44.53 rad/sec (7.09 Hz). Figure 4 also includes results obtained from the comparison model. As observed from the figure, the comparison model unconservatively predicts the buckling loads even if a fine mesh of 8 elements per member used. This is attributed to insufficient handling of distributed mass effects and nonlinear geometric effects and hence, more elements per member needed for better comparisons.

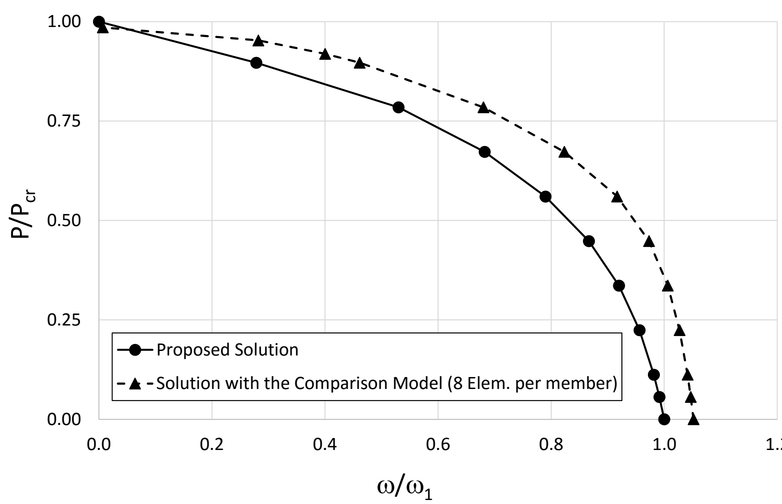


Figure 4 Buckling Load-Frequency Interaction Curve for Moment Frame

6.2. Dynamic Analysis of Two-Bay Moment Frame

Dynamic response of a two-bay moment frame subjected to vertical and lateral loads are studied in this example. The problem definition is given in Fig. 5 . The moment frame is braced with viscous dampers but they are only considered for damping lateral deflections otherwise they do not provide any lateral stiffness. It is also assumed that the dampers are massless. Sections W10x33 and W10x60 are selected for columns and beams, respectively (i.e., $I = I_{col} = 7.12$ x

10^7 mm^4 (171 in^4) and $I_{beam} = 1.42$ x 10^8 mm^4 (341 in^4)). In addition, the following numerical values are selected for this example: L = 3048mm (10ft), Po = 44.48 kN (10 kips) and E=200,000 MPa (29,000 ksi). On the assumption that the axial deformations are neglected, an inertia term of " $-2m_{beam} L_{beam} \omega^2$ " needs to be added to assembled stiffness matrix for horizontal degree of freedom.

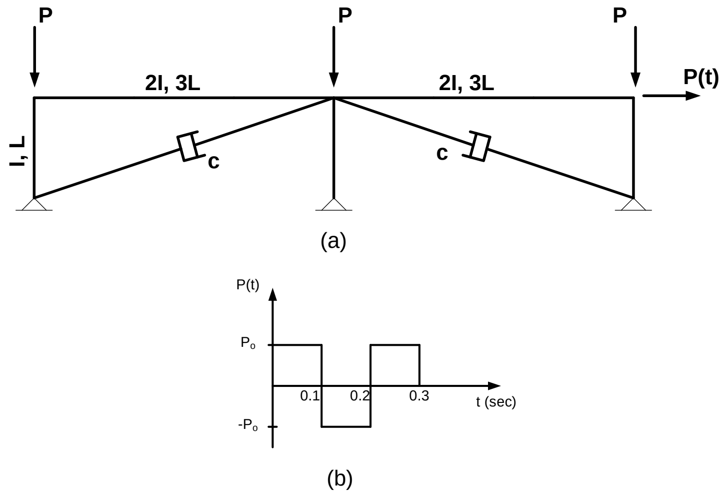


Figure 5 Problem Definition of a Two-Bay Moment Frame

Buckling load of the frame (i.e., $P_{cr} = \alpha EI / L^2$) is calculated in a similar way explained in Example 1. Table 4 compares the present solution result with other solutions.

Table 4: Buckling Load Factor of Two-bay Moment Frame

	α
Present Solution	1.690
Reference (Galambos, et. al., 2008)	1.686
FE++2015 (2 Elem. per member)	1.692

Dynamic analysis of the frame is carried out in frequency domain. This requires repetitive solution of Eq. (45) for values of ω. In most cases, external dynamic loads are expressed in terms of time and they must be transformed to frequency domain. Similarly, calculated results in frequency domain (such as $\Delta(\omega)$) can be inverse transformed to the time domain. An efficient numerical solution for transforming between frequency and time domain is essential. Fast Fourier Transform (FFT) and its inverse form (iFFT) are used for this purpose (Press et. al., 1992). Special care must be given to FFT parameters such as cut-of-frequency (ω_c) and number of sampling points (N). The cut-of-frequency determines highest frequency which can be represented in the solution. Beyond this value, it is assumed that the solution is negligible. Number of sampling points determines number of time or frequency increments at which solution is calculated. Once ω_c is chosen, time increment is simply $\Delta_t = \pi / \omega_c$. For this problem, $\Delta_t = 0.001$ sec and $N = 2^{12} = 4096$ are selected (i.e., $\omega_c = 3142$ rad/s).

The force in the damper is expressed as

$$F(t) = c\frac{du}{dt} \tag{49}$$

in which the constant term c is viscous damper coefficient and, u and du/dt are the axial displacement and velocity in the damper, respectively. After applying Fourier transform to Eq. (49), the following is obtained:

$$F(t) = c\,\omega\, i\, u(\omega) \tag{50}$$

The viscous damper element is treated as a truss-like element and its frequency-dependent stiffness matrix becomes

$$K_d = \begin{bmatrix} c\,\omega i & -c\,\omega i \\ -c\,\omega i & c\,\omega i \end{bmatrix} \tag{51}$$

The viscous damping coefficient (c) is selected as 4.025 N-sec/mm (0.023 kip-sec/in) and this is approximately equivalent to 5 percent Rayleigh damping.

Figure 6 shows the steps for frequency domain solution. Externally applied loads are calculated at each time increment ($P(t_i)$) and this data is transformed to frequency domain by the help of FFT. After this transformation, the external loads are expressed in terms of frequency intervals ($P(\omega_i)$). Once stiffness matrix is constructed for ω_i, it is solved to obtain displacements in frequency domain ($\Delta(\omega_i)$). This process is repeated for $\omega_i = i\,\Delta\omega$, $i = 1,2,\ldots,N$ and note that $\Delta\omega = 2\omega_c / N$. Finally, displacements in frequency domain are inverse transformed back to time domain by iFFT.

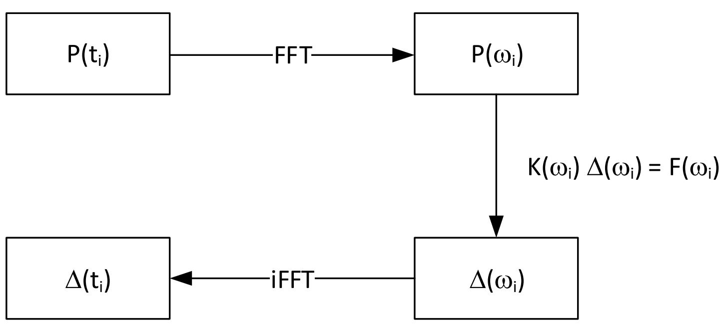
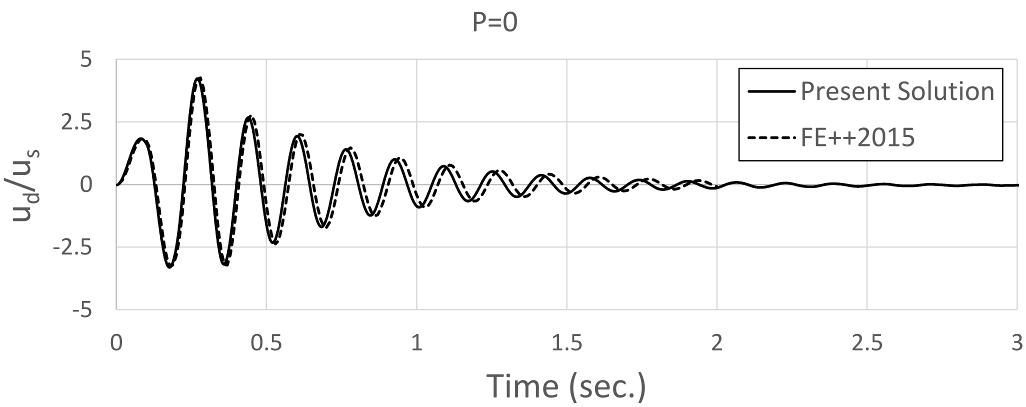


Figure 6 Steps for Frequency Domain Solution

For axial loads of $P = 0$ and $P_{cr} = 1.2\,EI/L^2$, lateral displacement of the frame is calculated and these results are portrayed graphically in Fig 7. Note that horizontal displacement is normalized with a static displacement of $u_s =$ 16.13 mm (0.635 in), which is obtained from an analysis with dynamic effects ignored.

The same example is also solved in time domain with Newmark-β (average acceleration) time-integration method (FE++2015). This solution models beams and columns with 4 elements per member. Time increment of $\Delta t = 0.001$ sec. is selected and classical Rayleigh damping with 0.05 for 1st and 3rd eigenmodes is included in the solution.



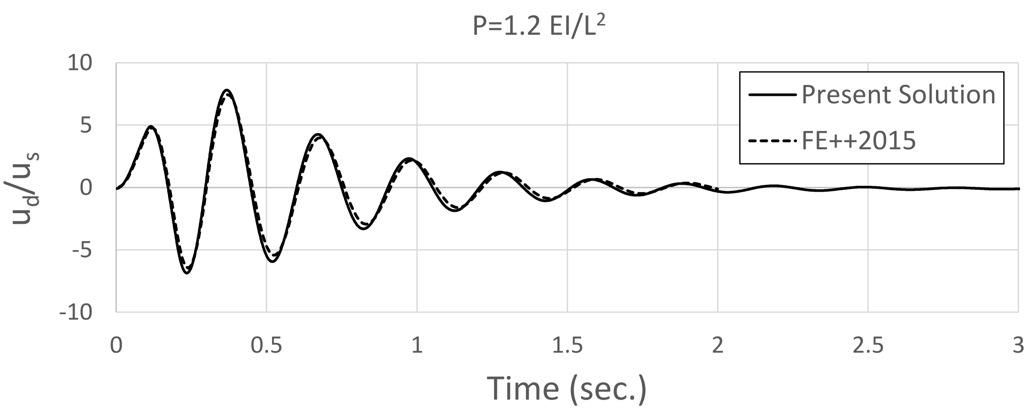


Figure 7 Time History Response of Lateral Displacement of the Moment Frame

7. Conclusions

Dynamics of a beam-column element resting on an elastic Winkler foundation and subjected to axial load is investigated in this study. An exact solution of dynamic equilibrium equations is pursued. Exact shape functions are derived and they are used to obtain closed forms of frequency-dependent exact stiffness terms. A total of four cases are identified and each case is individually studied. Distributed mass effects and geometrically nonlinear effects are directly included in the stiffness terms. Dynamic equilibrium equations are not only satisfied at element boundary nodes but also they are fulfilled within element. For this reason, only one element per member suffices to obtain accurate results as demonstrated by the examples. The proposed solution is also extended to other engineering problems for which exact stiffness terms are derived and provided in the Appendices. It is demonstrated that the proposed solution can be used in a typical finite element analysis framework such that more complex models can be addressed without any difficulty.

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Appendix A: Frequency-dependent Exact Shape Functions for Case 1

$$N_1 = \frac{\begin{pmatrix} \beta Cos[\gamma x](-\gamma + \beta Sin[L\gamma]) + \\ \beta Cosh[L\gamma](\gamma Cos[(L-x)\beta] + \gamma Cos[L\beta]Cosh[\gamma x] - \beta Sin[L\beta]Sinh[\gamma x]) - \\ \gamma(\beta Cos[\beta x] + Sinh[L\gamma](\gamma Sin[(L-x)\beta] + \beta Cos[L\beta]Sinh[\gamma x])) \end{pmatrix}}{2\beta\gamma(-1 + Cos[L\beta]Cosh[L\gamma]) + (\beta-\gamma)(\beta+\gamma)Sin[L\beta]Sinh[L\gamma]} \quad \text{(A.1)}$$

$$N_2 = \frac{\begin{pmatrix} -\gamma\, Sin[\beta\, x] + \beta\left(Cos[(L-x)\beta] - Cos[L\,\beta]Cosh[\gamma\, x]\right)Sinh[L\,\gamma] - \\ (\beta + \gamma\, Sin[L\,\beta]Sinh[L\,\gamma])Sinh[\gamma\, x] + \\ Cosh[L\,\gamma](\gamma\, Cosh[\gamma\, x]Sin[L\,\beta] - \gamma\, Sin[(L-x)\beta] + \beta\, Cos[L\,\beta]Sinh[\gamma\, x]) \end{pmatrix}}{2\,\beta\,\gamma\left(-1 + Cos[L\beta]Cosh[L\gamma]\right) + (\beta-\gamma)(\beta+\gamma)Sin[L\beta]Sinh[L\gamma]} \tag{A.2}$$

$$N_3 = \frac{\begin{pmatrix} -\gamma\begin{pmatrix} \beta(Cos[(L-x)\beta] - Cos[L\,\beta]Cosh[\gamma\, x] + Cosh[L\,\gamma](-Cos[\beta\, x] + Cosh[\gamma\, x])) + \\ \gamma\, Sin[\beta\, x]Sinh[L\,\gamma] \end{pmatrix} + \\ \beta\left(\beta\, Sin[L\,\beta] + \gamma\, Sinh[L\,\gamma]\right)Sinh[\gamma\, x] \end{pmatrix}}{2\,\beta\,\gamma\left(-1 + Cos[L\beta]Cosh[L\gamma]\right) + (\beta-\gamma)(\beta+\gamma)Sin[L\beta]Sinh[L\gamma]} \tag{A.3}$$

$$N_4 = \frac{\begin{pmatrix} \gamma\, Sin[(L-x)\beta] + \gamma\, Cosh[L\,\gamma]Sin[\beta\, x] - \beta\, Cos[\beta\, x]Sinh[L\gamma] + \\ Cos[\gamma\, x](-\gamma\, Sin[L\,\beta] + \beta\, Sinh[L\gamma]) + \beta\left(Cos[L\beta] - Cosh[L\gamma]\right)Sinh[\gamma\, x] \end{pmatrix}}{2\,\beta\,\gamma\left(-1 + Cos[L\beta]Cosh[L\gamma]\right) + (\beta-\gamma)(\beta+\gamma)Sin[L\beta]Sinh[L\gamma]} \tag{A.4}$$

Appendix B: Dynamic Stiffness Terms for Beam-Column on Elastic Foundation: Case 1

$$K_{11} = -\frac{EI\,\beta\,\gamma\left(\beta^2+\gamma^2\right)\left(\beta\, Cosh[L\gamma]Sin[L\beta] + \gamma\ Cos[L\beta]Sinh[L\gamma]\right)}{-2\,\beta\,\gamma + 2\,\beta\,\gamma\, Cos[L\,\beta]Cosh[L\gamma] + \left(\beta^2-\gamma^2\right)Sin[L\beta]Sinh[L\gamma]} \tag{B.1}$$

$$K_{12} = \frac{EI\,\beta\,\gamma\left(-\beta^2+\gamma^2+\left(\beta^2-\gamma^2\right)Cos[L\beta]Cosh[L\gamma] - 2\beta\,\gamma\, Sin[L\beta]Sinh[L\gamma]\right)}{-2\,\beta\,\gamma + 2\,\beta\,\gamma\, Cos[L\,\beta]Cosh[L\gamma] + \left(\beta^2-\gamma^2\right)Sin[L\beta]Sinh[L\gamma]} \tag{B.2}$$

$$K_{13} = \frac{EI\,\beta\,\gamma\left(\beta^2+\gamma^2\right)\left(\beta\, Sin[L\beta] + \gamma\ \ Sinh[L\gamma]\right)}{-2\,\beta\,\gamma + 2\,\beta\,\gamma\, Cos[L\,\beta]Cosh[L\gamma] + \left(\beta^2-\gamma^2\right)Sin[L\beta]Sinh[L\gamma]} \tag{B.3}$$

$$K_{14} = \frac{EI\,\beta\,\gamma\left(\beta^2+\gamma^2\right)\left(Cos[L\beta] - Cosh[L\gamma]\right)}{-2\,\beta\,\gamma + 2\,\beta\,\gamma\, Cos[L\,\beta]Cosh[L\gamma] + \left(\beta^2-\gamma^2\right)Sin[L\beta]Sinh[L\gamma]} \tag{B.4}$$

$$K_{22} = \frac{EI\ \left(\beta^2+\gamma^2\right)\left(-\gamma\, Cosh[L\gamma]Sin[L\beta] + \beta\, Cos[L\beta]Sinh[L\gamma]\right)}{-2\,\beta\,\gamma + 2\,\beta\,\gamma\, Cos[L\,\beta]Cosh[L\gamma] + \left(\beta^2-\gamma^2\right)Sin[L\beta]Sinh[L\gamma]} \tag{B.5}$$

$$K_{23} = -\frac{EI\,\beta\,\gamma\left(\beta^2+\gamma^2\right)\left(Cos[L\beta] - Cosh[L\gamma]\right)}{-2\,\beta\,\gamma + 2\,\beta\,\gamma\, Cos[L\,\beta]Cosh[L\gamma] + \left(\beta^2-\gamma^2\right)Sin[L\beta]Sinh[L\gamma]} \tag{B.6}$$

$$K_{24} = -\frac{EI\ \left(\beta^2+\gamma^2\right)\left(-Sin[L\beta] + \beta\, Sinh[L\gamma]\right)}{-2\,\beta\,\gamma + 2\,\beta\,\gamma\, Cos[L\,\beta]Cosh[L\gamma] + \left(\beta^2-\gamma^2\right)Sin[L\beta]Sinh[L\gamma]} \tag{B.7}$$

$$K_{33} = K_{11} \qquad K_{34} = -K_{12} \qquad K_{44} = K_{22} \tag{B.8}$$

Appendix C: Dynamic Stiffness Terms for Beam-Column on Elastic Foundation: Case 3

$A < 2\sqrt{B}$:

$$K_{11} = \frac{2\,EI\,\xi\,\eta\left(\xi^2+\eta^2\right)\left(\eta\,Sin[2L\xi]+\xi\,Sinh[2L\eta]\right)}{-\xi^2-\eta^2+\eta^2\,Cos[2L\xi]+\xi^2 Cosh[2L\eta]} \tag{C.1}$$

$$K_{22} = \frac{EI\;\left(\xi^2+\eta^2\right)\left(-\xi^2+\eta^2-\eta^2\,Cos[2L\xi]+\xi^2\,Cosh[2L\eta]\right)}{-\xi^2-\eta^2+\eta^2\,Cos[2L\xi]+\xi^2 Cosh[2L\eta]} \tag{C.2}$$

$$K_{13} = \frac{4\,EI\,\xi\,\eta\left(\xi^2+\eta^2\right)\left(\eta\,Cosh[L\eta]Sin[L\xi]+\xi\,Cos[L\xi]Sinh[L\eta]\right)}{-\xi^2-\eta^2+\eta^2\,Cos[2L\xi]+\xi^2 Cosh[2L\eta]} \tag{C.3}$$

$$K_{14} = \frac{4\,EI\,\xi\,\eta\left(\xi^2+\eta^2\right)Sin[L\xi]Sinh[L\eta]}{-\xi^2-\eta^2+\eta^2\,Cos[2L\xi]+\xi^2 Cosh[2L\eta]} \tag{C.4}$$

$$K_{22} = \frac{2\,EI\,\xi\,\eta\;\left(-\eta\,Sin[2L\xi]+\xi\,Sinh[2L\eta]\right)}{-\xi^2-\eta^2+\eta^2\,Cos[2L\xi]+\xi^2 Cosh[2L\eta]} \tag{C.5}$$

$$K_{23} = \frac{4\,EI\,\xi\,\eta\left(\xi^2+\eta^2\right)Sin[L\xi]Sinh[L\eta]}{-\xi^2-\eta^2+\eta^2\,Cos[2L\xi]+\xi^2 Cosh[2L\eta]} \tag{C.6}$$

$$K_{24} = \frac{4\,EI\,\xi\,\eta\;\left(-\eta\,Cosh[L\eta]Sin[L\xi]+\xi\,Cos[L\xi]Sinh[L\eta]\right)}{-\xi^2-\eta^2+\eta^2\,Cos[2L\xi]+\xi^2 Cosh[2L\eta]} \tag{C.7}$$

$$K_{33} = K_{11} \qquad K_{34} = -K_{12} \qquad K_{44} = K_{22} \tag{C.8}$$

$A = 2\sqrt{B}$:

$$K_{11} = \frac{2\,EI\,\beta^3\;\left(2L\beta+Sin[2L\beta]\right)}{-1+2L^2\beta^2+Cos[2L\beta]} \qquad K_{12} = \frac{EI\,\beta^2\;\left(1+2L^2\beta^2-Cos[2L\beta]\right)}{-1+2L^2\beta^2+Cos[2L\beta]} \tag{C.9}$$

$$K_{13} = -\frac{4\,EI\,\beta^3\;\left(L\beta\,Cos[L\beta]+Sin[L\beta]\right)}{-1+2L^2\beta^2+Cos[2L\beta]} \qquad K_{14} = \frac{4\,EI\,L\beta^3\;Sin[L\beta]}{-1+2L^2\beta^2+Cos[2L\beta]} \tag{C.10}$$

$$K_{22} = \frac{2\,EI\,\beta\;\left(2L\beta-Sin[2L\beta]\right)}{-1+2L^2\beta^2+Cos[2L\beta]} \qquad K_{23} = \frac{4\,EI\,L\beta^3\;Sin[L\beta]}{-1+2L^2\beta^2+Cos[2L\beta]} \tag{C.11}$$

$$K_{24} = -\frac{4\,EI\,\beta\;\left(L\beta\,Cos[L\beta]-Sin[L\beta]\right)}{-1+2L^2\beta^2+Cos[2L\beta]} \tag{C.12}$$

$$K_{33} = K_{11} \qquad K_{34} = -K_{12} \qquad K_{44} = K_{22} \tag{C.13}$$

$A > 2\sqrt{B}$:

$$K_{11} = -\frac{EI\,\beta(\beta-\gamma)\,\gamma(\beta+\gamma)(\beta\,Cos[L\gamma]Sin[L\beta]-\gamma\,Cos[L\beta]Sin[L\gamma])}{2\beta\gamma(-1+Cos[L\beta]Cos[L\gamma])+(\beta^2+\gamma^2)Sin[L\beta]Sin[L\gamma]} \tag{C.14}$$

$$K_{12} = \frac{EI\,\beta\gamma\,((\beta^2+\gamma^2)(-1+Cos[L\beta]Cos[L\gamma])+2\beta\gamma\,Sin[L\beta]Sin[L\gamma])}{2\beta\gamma(-1+Cos[L\beta]Cos[L\gamma])+(\beta^2+\gamma^2)Sin[L\beta]Sin[L\gamma]} \tag{C.15}$$

$$K_{13} = \frac{EI\,\beta(\beta-\gamma)\,\gamma(\beta+\gamma)(\beta\,Sin[L\beta]-\gamma\,Sin[L\gamma])}{2\beta\gamma(-1+Cos[L\beta]Cos[L\gamma])+(\beta^2+\gamma^2)Sin[L\beta]Sin[L\gamma]} \tag{C.16}$$

$$K_{14} = \frac{EI\,\beta(\beta-\gamma)\,\gamma(\beta+\gamma)(Cos[L\beta]-Cos[L\gamma])}{2\beta\gamma(-1+Cos[L\beta]Cos[L\gamma])+(\beta^2+\gamma^2)Sin[L\beta]Sin[L\gamma]} \tag{C.17}$$

$$K_{22} = \frac{EI\,(\beta-\gamma)(\beta+\gamma)(-\gamma\,Cos[L\gamma]Sin[L\beta]+\beta\,Cos[L\beta]Sin[L\gamma])}{2\beta\gamma(-1+Cos[L\beta]Cos[L\gamma])+(\beta^2+\gamma^2)Sin[L\beta]Sin[L\gamma]} \tag{C.18}$$

$$K_{23} = -\frac{EI\,\beta(\beta-\gamma)\,\gamma(\beta+\gamma)(Cos[L\beta]-Cos[L\gamma])}{2\beta\gamma(-1+Cos[L\beta]Cos[L\gamma])+(\beta^2+\gamma^2)Sin[L\beta]Sin[L\gamma]} \tag{C.19}$$

$$K_{24} = -\frac{EI\,(\beta-\gamma)\,(\beta+\gamma)(-\gamma\,Sin[L\beta]+\beta\,Sin[L\gamma])}{2\beta\gamma(-1+Cos[L\beta]Cos[L\gamma])+(\beta^2+\gamma^2)Sin[L\beta]Sin[L\gamma]} \tag{C.20}$$

$$K_{33} = K_{11} \qquad K_{34} = -K_{12} \qquad K_{44} = K_{22} \tag{C.21}$$

Appendix D: Static Stiffness Terms for Beam-Column

$$K_{11} = -\frac{EI\,\beta^3\,Cos\left[\frac{L\beta}{2}\right]}{L\beta\,Cos\left[\frac{L\beta}{2}\right]-2Sin\left[\frac{L\beta}{2}\right]} \qquad K_{12} = -\frac{EI\,\beta^2\,Sin\left[\frac{L\beta}{2}\right]}{L\beta\,Cos\left[\frac{L\beta}{2}\right]-2Sin\left[\frac{L\beta}{2}\right]} \tag{D.1}$$

$$K_{13} = \frac{EI\,\beta^3\,Cos\left[\frac{L\beta}{2}\right]}{L\beta\,Cos\left[\frac{L\beta}{2}\right]-2Sin\left[\frac{L\beta}{2}\right]} \qquad K_{14} = -\frac{EI\,\beta^2\,Sin\left[\frac{L\beta}{2}\right]}{L\beta\,Cos\left[\frac{L\beta}{2}\right]-2Sin\left[\frac{L\beta}{2}\right]} \tag{D.2}$$

$$K_{22} = \frac{EI\,\beta\left(Csc\left[\frac{L\beta}{2}\right]\right)^2(L\beta\,Cos[L\beta]-Sin[L\beta])}{-4+2\,L\beta\,Cot\left[\frac{L\beta}{2}\right]} \tag{D.3}$$

$$K_{23} = \frac{EI\,\beta^2\,Sin\left[\frac{L\beta}{2}\right]}{L\beta\,Cos\left[\frac{L\beta}{2}\right] - 2Sin\left[\frac{L\beta}{2}\right]} \tag{D.4}$$

$$K_{24} = \frac{EI\,\beta\left(Csc\left[\frac{L\beta}{2}\right]\right)^2\left(-L\beta + Sin[L\beta]\right)}{-4 + 2\,L\beta\,Cot\left[\frac{L\beta}{2}\right]} \tag{D.5}$$

$$\beta = \sqrt{\frac{P}{EI}} \tag{D.6}$$

$$K_{33} = K_{11} \qquad K_{34} = -K_{12} \qquad K_{44} = K_{22} \tag{D.7}$$

Appendix E: Dynamic Stiffness Terms for Beam

$$K_{11} = \frac{-\left(EI\,\eta^3\left(Cosh[L\eta]Sin[L\eta] + Cos[L\eta]Sinh[L\eta]\right)\right)}{-1 + Cos[L\eta]Cosh[L\eta]} \tag{E.1}$$

$$K_{12} = \frac{EI\,\eta^2\,Sin[L\eta]Sinh[L\eta]}{1 - Cos[L\eta]Cosh[L\eta]} \tag{E.2}$$

$$K_{13} = \frac{EI\,\eta^3\left(Sin[L\eta] + Sinh[L\eta]\right)}{-1 + Cos[L\eta]Cosh[L\eta]} \tag{E.3}$$

$$K_{14} = \frac{EI\,\eta^2\left(Cos[L\eta] - Cosh[L\eta]\right)}{-1 + Cos[L\eta]Cosh[L\eta]} \tag{E.4}$$

$$K_{22} = \frac{\left(EI\,\eta\left(-Cosh[L\eta]Sin[L\eta] + Cos[L\eta]Sinh[L\eta]\right)\right)}{-1 + Cos[L\eta]Cosh[L\eta]} \tag{E.5}$$

$$K_{23} = \frac{EI\,\eta^2\left(-Cos[L\eta] + Cosh[L\eta]\right)}{-1 + Cos[L\eta]Cosh[L\eta]} \tag{E.6}$$

$$K_{24} = \frac{EI\,\eta\left(Sin[L\eta] - Sinh[L\eta]\right)}{-1 + Cos[L\eta]Cosh[L\eta]} \tag{E.7}$$

$$\eta = \sqrt[4]{\frac{m\,\omega^2}{EI}} \tag{E.9}$$

$$K_{33} = K_{11} \qquad K_{34} = -K_{12} \qquad K_{44} = K_{22} \tag{E.10}$$