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Ten years of research on stability of thin-walled members revisited

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Abstract
This paper corresponds to the text version of the presentation delivered on April 11, 2018, at the SSRC Annual Stability Conference, in the context of the 2017 SSRC McGuire Award for Junior Researchers. The paper reviews the most relevant research in which the author has been involved, in the topic of stability of thin-walled members and held in the past ten years. In particular, the paper addresses three main areas: Generalized Beam Theory, geometrically exact beam formulations and structural design aspects.

1. Introduction

The next Sections describe the work carried out by the author and co-workers, in the last ten years, focusing on the most relevant findings. Section 2 describes the achievements in the field of Generalized Beam Theory (GBT), Section 3 addresses the research carried out concerning geometrically exact beam theories and Section 4 focuses on aspects related to structural design aspects. The paper closes in Section 5, with the concluding remarks.

The work reported in this paper has been mostly co-authored by Professor Dinar Camotim, of the Lisbon University, Portugal, whose teachings have left, unquestionably, a most valuable and indelible mark on the author. Other colleagues, research fellowship holders and students, namely of the GBT research group within the CERIS research center in Lisbon (see www.civil.ist.utl.pt/gbt/), which have participated in the research, are mentioned in the refences.

2. Generalized Beam Theory

GBT is a thin-walled bar theory that efficiently handles cross-section in-plane and out-of-plane deformation through the consideration of so-called “cross-section deformation modes”. This theory has been initially developed about 50 years ago by Richard Schardt (Schardt 1966, 1989; see also http://www.vtb.info/ for a list of publications up to 2004) and has been considerably developed ever since, particularly by the Lisbon-based research group (see www.civil.ist.utl.pt/gbt/ for the associated list of publications). Currently, GBT has been established as a very efficient tool to analyze thin-walled members, with significant advantages with respect to shell finite element models in the small-to-intermediate displacement range, namely due to its computational efficiency and modal decomposition features (the GBT

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deformation modes have a clear structural meaning). Moreover, it is quite straightforward to include/exclude specific effects and, in some cases, it is possible to obtain semi-analytical or even analytical solutions.

Most of the work carried out by the author falls in this field, namely in the development and application of new GBT formulations, as described next.

2.1 Deformation modes

The first step of a GBT analysis is the determination of the cross-section deformation modes. This is a key step that significantly affects the performance of the GBT member analysis. In particular, this step should include as many admissible constraints as possible (e.g., the Kirchhoff thin plate assumption, which is generally adopted), to subdivide and/or reduce the number of deformation modes, and thus, degrees-of-freedom (DOFs). The book by Schardt (1989) addressed the case of open unbranched cross-sections, in which case it is generally acceptable to assume null membrane shear strains (Vlasov assumption) and null membrane transverse extensions. For unicellular sections, Schardt included an additional torsional mode that involves a constant shear flow.

For arbitrary (flat-walled) cross-sections, the introduction of the constraints is much more complex. Open cross-sections complying with the Vlasov and null transverse extension assumptions were addressed by Möller (1982) and Dinis et al. (2006). Möller also examined the case of closed cross-sections with the null transverse extension assumption, following an approach similar to Vlasov's (1959). However, this approach is prone to shear locking and it does not retrieve the classic bending and torsion modes. This fact led the author to develop a method for arbitrary cross-sections that enables the introduction of membrane shear strains in selected walls, while keeping the Vlasov assumption in the remaining walls (Gonçalves et al. 2009). Later this approach was generalized to arbitrary constraints, such as rigid links and cross-bracings, as well as special cross-sections with walls that cannot undergo independent shearing (Gonçalves et al. 2010b). In this work, methods were provided to calculate the number of modes in each subset. Finally, a method for obtaining and hierarchizing the shear deformation modes for arbitrary cross-sections, including the torsion and cell shear flow modes, as well as formulas for calculating the number of modes in each subset, were provided in (Gonçalves et al. 2014a).

The allowance for membrane transverse extensions leads to more modes which have been used in several investigations concerning open sections (e.g., Miosga 1976, Silvestre & Camotim 2003) and circular tubes (Schardt 1985). These modes, together with the computational efficient approaches outlined in (Gonçalves et al. 2014a, Bebiano et al. 2015), were implemented in the GBTUL program (Bebiano et al. 2018b), available at www.civil.ist.utl.pt/gbt/ — see also (Bebiano et al. 2018a), which presents a set of relevant applications.

For illustrative purposes, Fig. 1 shows results concerning the first-order analysis of a simply supported beam with a twin trapezoidal cell cross-section and a length of 600 mm, subjected to a single half-wave sinusoidal vertical load (Garcea et al. 2016). The cross-section is discretized using 21 intermediate nodes and the material parameters are E = 210 GPa, ν = 0,3. The figure displays (i) the results obtained with GBT and a refined shell model, (ii) the GBT modal participations and (iii) the most relevant higher-order deformation modes.

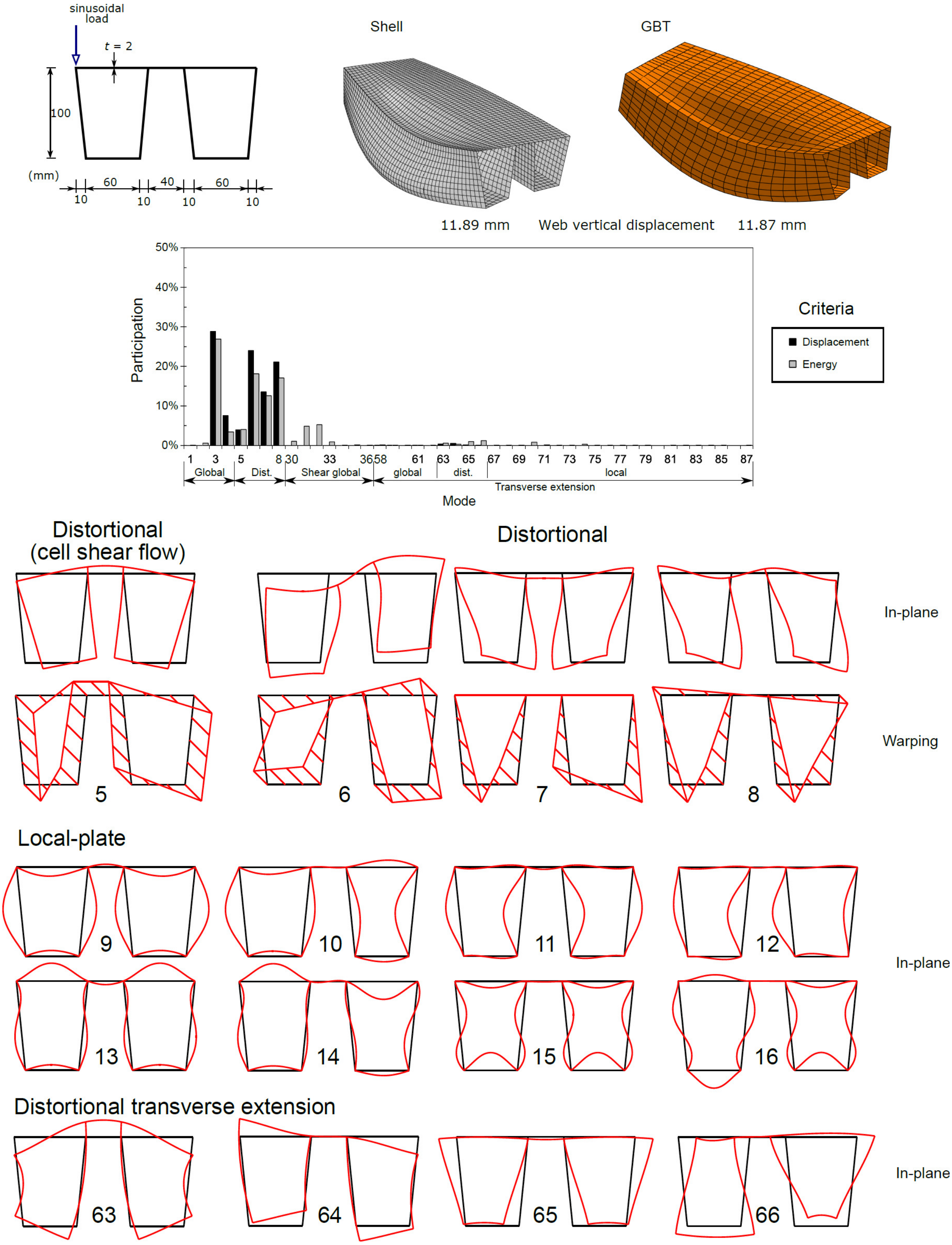


Figure 1: Deformation modes and first-order analysis results of a twin trapezoidal cell cross-section beam.

2.2 Non-linear materials

The first application of GBT to non-linear materials can be found in (Gonçalves & Camotim 2004), where J_2 (von Mises) incremental and deformation plasticity theories were implemented to calculate plastic bifurcation loads (linear stability analyses) of thin-walled columns made of strain hardening materials (aluminium and stainless steel), using semi-analytical solutions. Later, finite elements were developed to model general supports and loading, namely flexural members (Gonçalves & Camotim 2007). Finally, a comparison between GBT solutions, analytical solutions and results obtained with a special shell finite element formulation able to capture plastic bifurcation loads were presented in (Gonçalves et al. 2010a). It was found that the shell models yield results near those obtained with GBT deformation theory, due to boundary effects and pre-buckling deflections, which act as imperfections.

A first-order GBT formulation for elastoplastic materials was first presented in (Gonçalves & Camotim, 2011). The small-strain J_2 theory with associated flow rule and isotropic hardening was implemented in a finite element. Both stress-based and a shell-like stress resultant-based (Ilyushin yield function) formulations were developed. The latter is less accurate but has two major advantages: (i) it avoids through-thickness numerical integration, which is particularly expensive in materially non-linear problems, and (ii) enables constraining various stress/strain components to zero, as in the standard linear elastic GBT, thus lowering significantly the number of deformation modes required to achieve accurate results. The extension to the geometrically non-linear case was presented in (Gonçalves & Camotim 2012), where it was shown that, even though more deformation modes are needed to obtain accurate results, the shell-like stress resultant approach still provides accurate results while avoiding the expensive through-thickness numerical integration.

Steel plasticity and concrete non-linearity, including cracking and crushing, were combined in (Henriques et al. 2015). The concrete material model was specifically tailored to achieve computationally efficiency, by setting various stress and strain components to zero while still obtaining accurate results with a small number of deformation modes. A smeared fixed crack-type approach was adopted. For both efficiency and simplicity, separate constitutive laws were adopted for the longitudinal normal stresses and shear stresses. The former relate to the longitudinal strains through an uniaxial law, without tensile strength and a non-linear compressive branch up to the peak stress. Unloading/reloading is elastic and, after the peak, a linear softening branch is adopted with a mesh-adjusted tangent modulus, to mitigate mesh-dependency issues. For the shear stresses, a non-linear elastic relation was adopted, with a linear branch having a shear modulus multiplied by a reduction factor for cracked concrete, up to a maximum stress, after which horizontal plateaus follow.

Even with the previous strategies, the computational efficiency of GBT finite elements essentially decreases with non-linearity, since many deformation modes are generally required to obtain accurate results. In addition, severe mode coupling occurs, as the modes involve displacements of the complete cross-section, leading to large and dense matrices. This contrasts with shell finite elements, which involve small matrices and lead to sparse global stiffness matrices. For this reason, a new and general approach for material and geometrical non-linearity was proposed in (Gonçalves & Camotim 2017b), where the finite element DOFs correspond to cross-section nodal displacements and rotations. This approach is equivalent to using an assembly of quadrilateral "wall" elements, which are similar to the classic Bogner-Fox-Schmit plate element, but with added quadratic membrane displacements, except for the longitudinal interpolation of the mid-line direction displacements, in which case Hermite cubic functions are employed. After the structural analysis is performed, a post-processing technique is used to recover the GBT deformation mode participations. This approach can easily handle discrete variations of the wall thickness in the longitudinal direction, including holes.

Figure 2 displays the results of a geometrically and materially (elastic-perfectly plastic) non-linear analysis of a 300 mm long simply supported and uniformly compressed column with a lipped channel section. The column has a critical-mode (distortional) imperfection, with a 1 mm mid-span lip lateral displacement. Due to the problem double symmetry, only a quarter of the column is modelled. It is observed that the stress resultant-based approach yields fairly accurate results and that the node-based approach is the most accurate one for large displacements, while leading to a much sparser element tangent stiffness matrix (these matrices correspond to a typical iteration step).

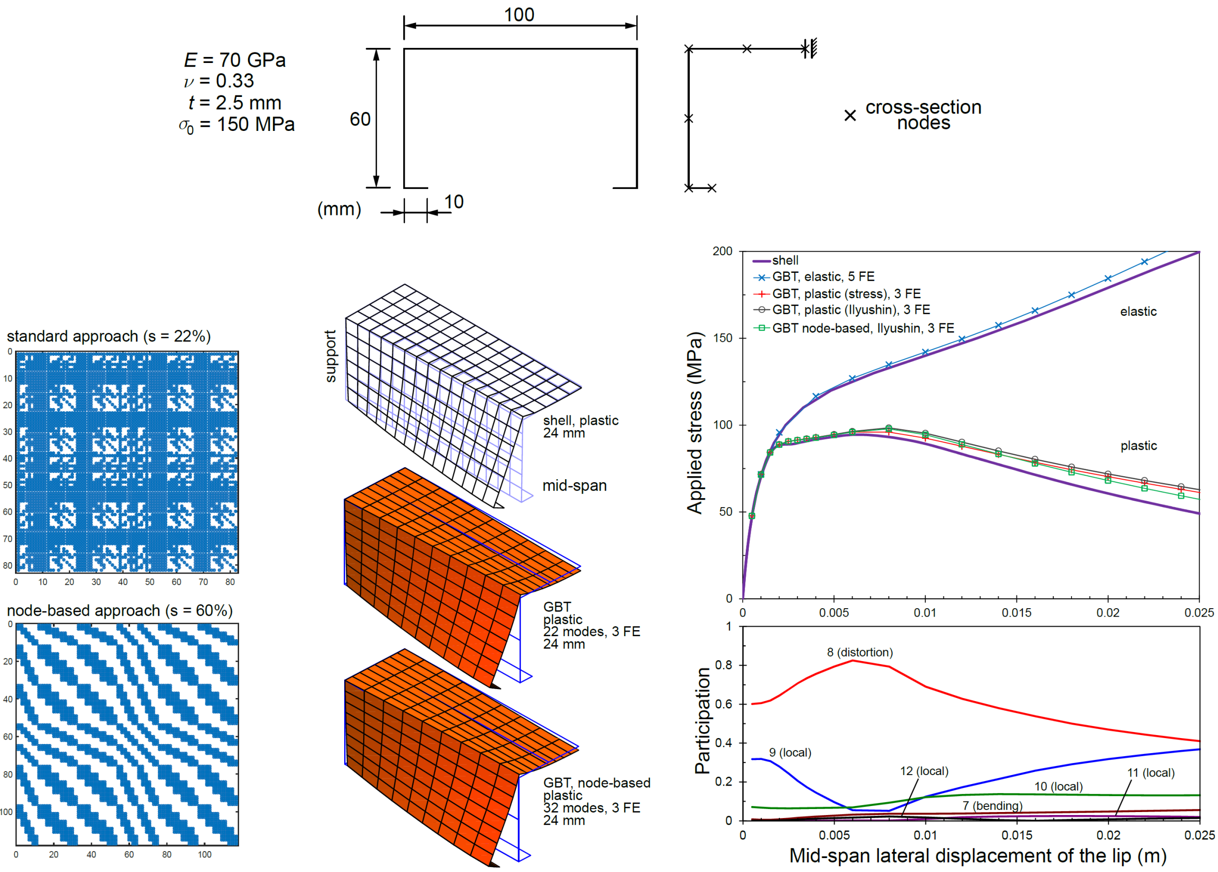


Figure 2: Elastoplastic collapse of a simply supported lipped channel column.

2.3 Regular convex polygonal tubes

Steel tubes with thin-walled single-cell regular convex polygonal cross-section (RCPS) enjoy widespread application in the steel construction industry, namely in towers and masts. The first-order, buckling (bifurcation) and undamped free vibration behavior of these tubes has been investigated in (Gonçalves & Camotim 2013a-c; 2014). New light on the subject was shed by developing a computational efficient GBT specialization for RCPS, which made it possible to (i) identify a set of fully orthogonal cross-section deformation modes, (ii) draw meaningful conclusions concerning the structural behavior of these tubes and also (iii) derive analytical and semi-analytical formulae. This GBT specialization takes advantage of the fact that RCPS have rotational symmetry of order equal to the number of walls, leading to real symmetric circulant GBT matrices for each mode set. These matrices share the same eigenvectors and thus can be fully diagonalized within each set in a very efficient way (with the exception of one local mode set). Since the equation system is much more uncoupled that in the standard GBT case, namely it is coupled by blocks, its solution is much more efficient and analytical solutions can be obtained in a wide range of problems. Furthermore, the eigenvalues appear in pairs, meaning that duplicate solutions exist. Figure 3 shows the shapes of the natural Vlasov distortional modes for RCPS with 4 to 8 walls. All modes appear in pairs, except for the last mode for an even number of walls.

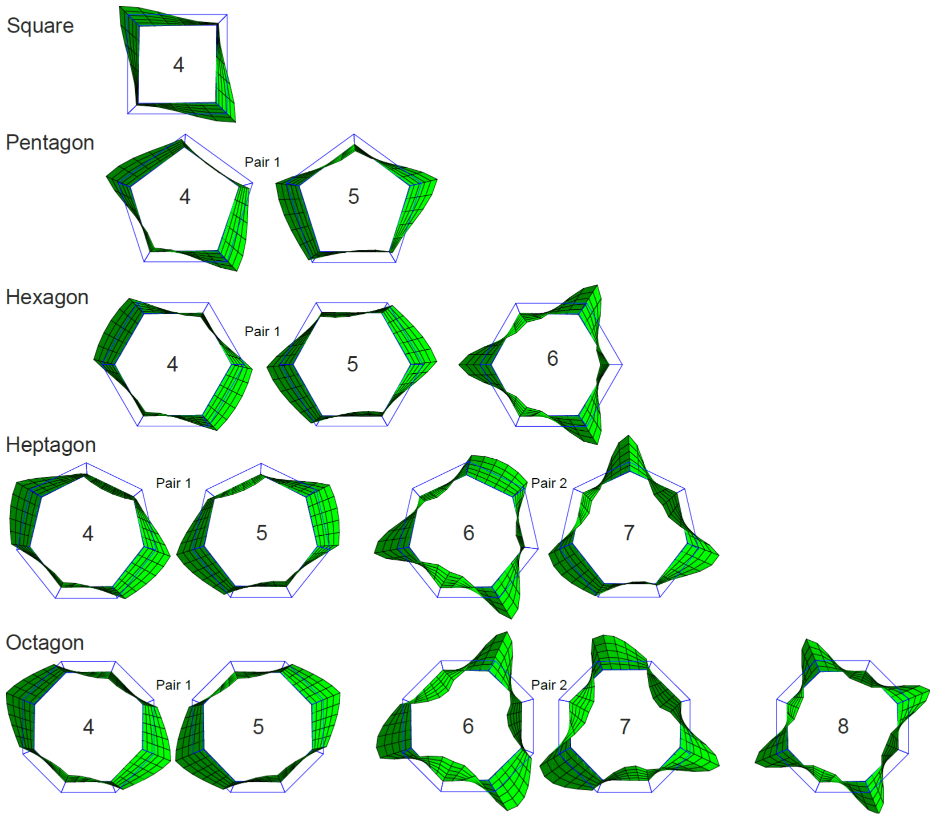


Figure 3: Shapes of the natural Vlasov distortional warping modes.

The local/distortional/extensional/global buckling behavior of RCPS tubes under compression was addressed in (Gonçalves & Camotim 2013b). Both analytical and numerical results were provided and it was shown that the local and distortional buckling modes appear in pairs. Furthermore, the parameter ranges for which each pure mode is critical were obtained. It was observed that distortional buckling may be significantly influenced by shear deformation and, to a smaller extent, by local-plate modes. Figure 4 shows (a) a typical signature curve, (b) the 2D mode space associated with the second distortional mode space for a tube with 20 walls and (c) the parameter ranges associated with local/distortional critical buckling.

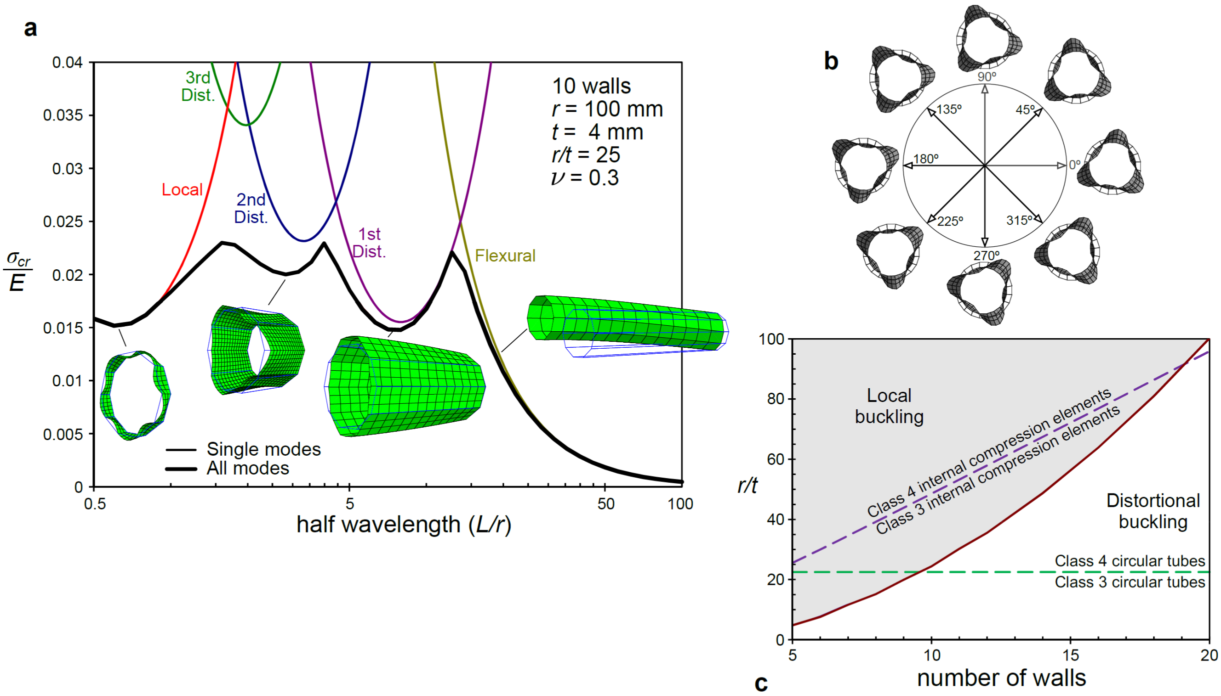


Figure 4: Simply supported compressed tubes: (a) "signature"/individual mode curves and buckling mode shapes, (b) distortional 2D buckling mode space for RCPS with 20 walls and (c) parameter ranges associated with local/distortional critical buckling.

For tubes under pure bending it was shown that lower local buckling stresses are obtained when the bending axis is parallel to a cross-section wall. However, as shown in Fig. 5, local buckling with non-null displacements of the wall junctions may occur as the number of walls increases (namely above 10). This figure also shows that the classic formula for circular tubes (the dashed lines) can be used to estimate the buckling stresses for RCPS tubes with non-null displacements of the wall junctions.

The case of pure torsion was also addressed. Besides pure local/distortional buckling, the influence of shear deformation and mode coupling was investigated. In general terms, the critical stress decreases with increasing member length, until a plateau is reached, which means that the calculation of the minimum critical load using shell element models normally requires many DOFs. With GBT, although several finite elements are needed, the GBT matrices are either diagonal or block diagonal with small blocks, meaning that the buckling loads may be calculated by inspecting each block separately, leading to significant computational savings. Furthermore, duplicate buckling modes are found.

For pure local buckling, the corresponding mode is similar to that observed for a simply supported rectangular plate under pure shear, although the nodal lines must form a continuous spiral along the length and thus the buckling coefficient (k) is higher than 5.336. Fig. 6 shows one of the duplicate modes for various tubes with length/wall width (L/b) ratios equal to 10 (thus leading to buckling stresses close to the minima). These results were obtained with five intermediate nodes in each wall and 10 to 15 GBT-based beam finite elements.

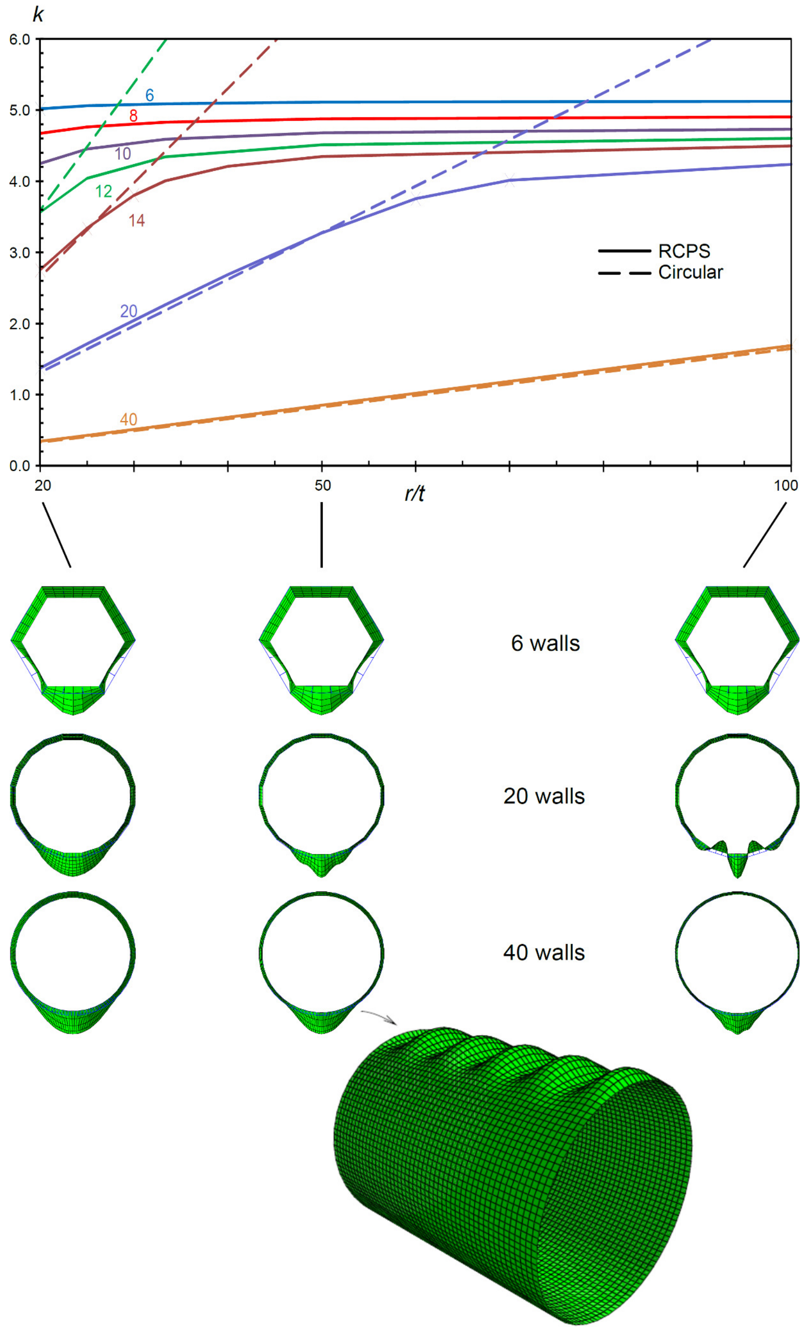


Figure 5: Local buckling of simply supported RCPS tubes under bending: buckling coefficients and mode shapes.

For pure distortional buckling, analytical expressions for the GBT geometric matrix components were obtained and it was concluded that the buckling mode can be viewed as a helix in the 2D mode space pertaining to the corresponding distortional deformation mode pair (see Fig. 7). Moreover, the first distortional mode pair is critical for infinitely long tubes and analytical expressions for the minimum critical loads were obtained by assuming that the helix has constant pitch and amplitude along the tube. It was demonstrated that the influence of shear deformation is negligible and that local/distortional coupling is only significant for RCPS with five walls.

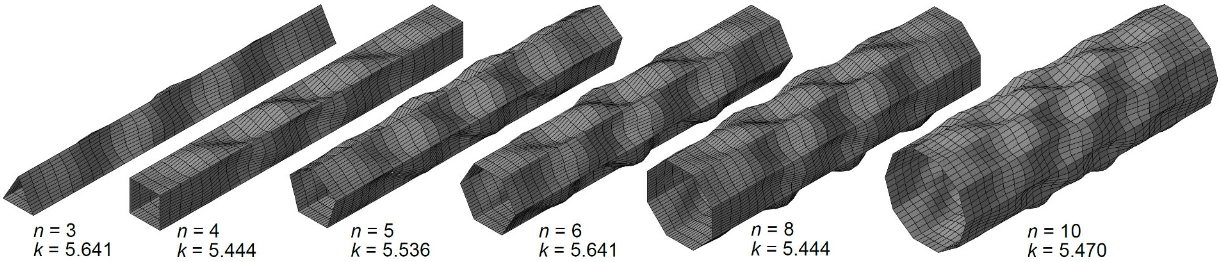


Figure 6: Pure local buckling of RCPS tubes under uniform torsion (n is the number of walls).

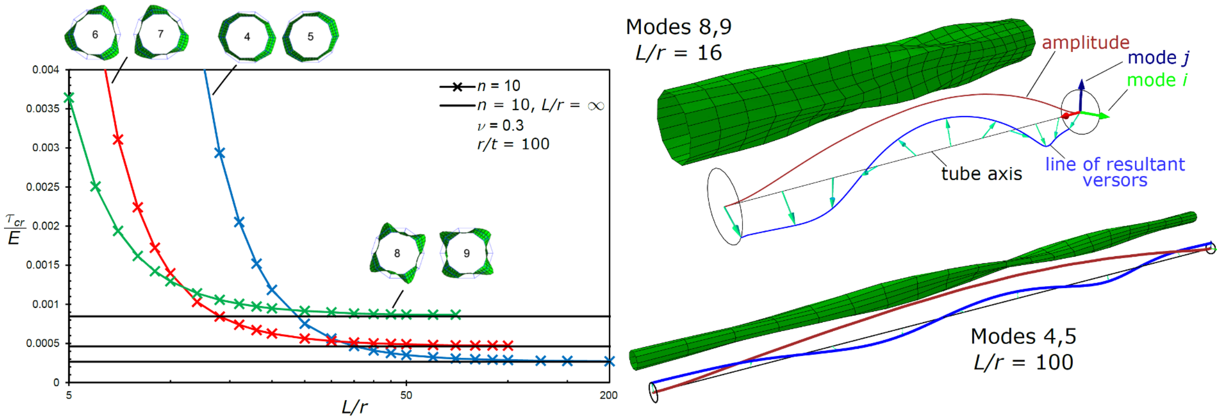


Figure 7: Pure distortional buckling for uniform torsion: critical stresses and buckling mode shapes.

Finally, the undamped free vibration behavior of simply supported thin-walled RCPS tubes was investigated in (Gonçalves & Camotim 2015). As in the previous analyses, the GBT specialization for RCPS made it possible to achieve new results. The vibration behaviors associated with each mode type (local-plate, extensional – transverse and longitudinal –, distortional and torsional) were first addressed separately. Analytical solutions were derived for each case, duplicate solutions were identified and the influence of the various parameters was investigated. Then, multi-mode coupling was addressed, including shear deformation. The frequency map was obtained for pentagonal cross-sections and attention was paid to vibration modes associated with single solutions involving peculiar local-torsional-extensional mode interactions. Finally, the fundamental frequencies and the associated vibration modes were examined and characterized. It was found that the local, distortional and bending modes are the most relevant, but shear deformation also plays a key role, particularly in the transition between local-distortional modes if the radius-to-thickness ratio is low and the number of walls is high.

The peculiarities of RCPS led to the investigation of the mechanics of (i) vibration of multi-cell beams (Gonçalves et al. 2015) and (ii) distortion in symmetric and periodic open cross-sections (Gonçalves & Camotim 2015). For open sections with reflectional symmetry, $n\times n$ symmetric centro-symmetric GBT matrices are obtained, with $\lceil n/2 \rceil$ (smallest integer $\geq n/2$) symmetric and $\lfloor n/2 \rfloor$ (largest integer $\leq n/2$) anti-symmetric eigenvectors which can be used to break down the problem. For rotational symmetry, the GBT matrices are symmetric block circulant and may be block-diagonalized, as in the case of RCPS. The distortional modes are then computed from each individual diagonal block, which is very efficient from a computational point of view, and mode pairs sharing the same eigenvalue are obtained for the double-size blocks.

Periodic sections can be generated through either translation or glide reflection of a "cell", along a straight line. For open sections, the GBT modal matrices can be efficiently assembled using only the matrices of the repeating cell. For infinite periodic sections, the use of periodic boundary conditions leads to symmetric block circulant finite-dimension matrices. For a given mode wavelength, the GBT matrices can be assembled from those pertaining to the cell and, subsequently, block diagonalized. As in the case of rotational symmetry, the distortional modes may be computed from each diagonal block, individually, and mode pairs are retrieved for the double-size blocks. Fig. 8 concerns an infinite periodic cross-section and shows the first distortional deformation mode space associated with a period of 5 cells.

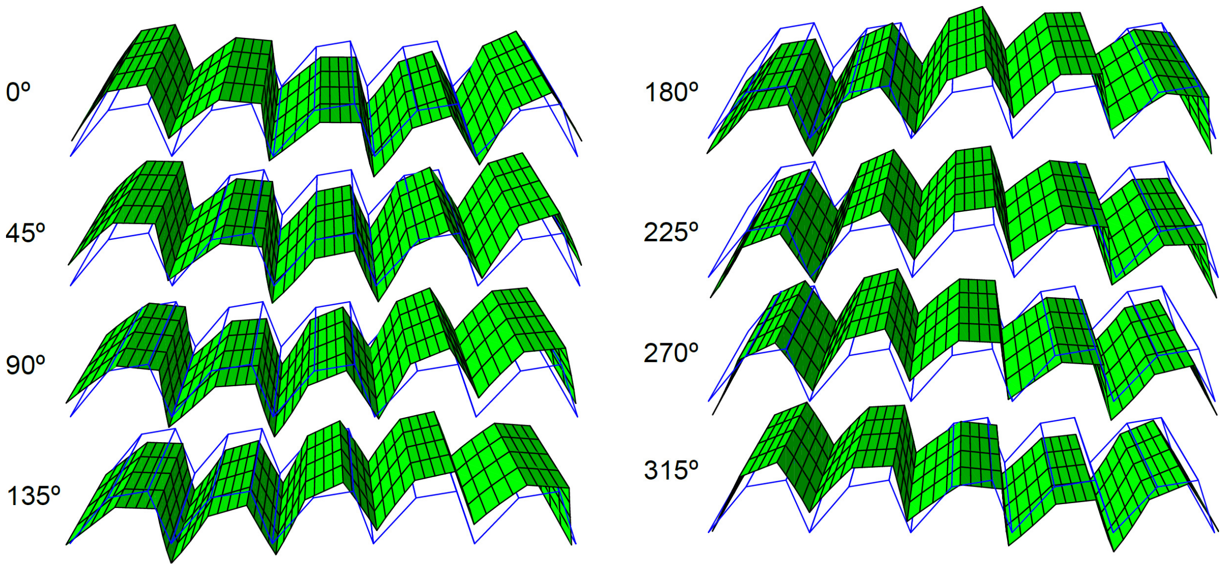


Figure 8: Distortional deformation mode pair space for a periodic cross-section (period of 5 cells).

2.4 Steel concrete composite beams

The first paper that applies GBT in the field of steel-concrete composite beams and bridges is (Gonçalves & Camotim 2010). In this paper, only linear elastic material behavior was considered, but several new effects, such as shear lag, shear connection flexibility and the presence of diaphragms, were considered.

In (Henriques et al. 2015) an efficient physically non-linear GBT-based beam finite element was proposed, aiming at capturing the global behavior, up to collapse, of wide flange steel and steel-concrete composite beams. Reinforced concrete non-linear behavior was introduced and combined with shear lag effects and steel beam plasticity. It should be noted that capturing physically non-linear shear lag with beam elements is rather challenging, particularly for very wide flanges, since the neutral surface is invariably located in the flange and the stresses vary rapidly across its thickness. The formulation thus was developed by making an appropriate trade-off between accuracy and computational efficiency, taking advantage of the inherent characteristics of GBT and aiming at simplicity. In particular, the stress and strain fields were appropriately constrained to limit the number of admissible deformation modes, in order to allow membrane shear deformation in relevant cross-section zones only (wide flanges and steel girder web) and employ simple material models for concrete and steel. With this element very accurate load-displacement paths, up to collapse, are obtained with a very small computational cost. With an Intel Core i7 CPU @ 2.10 GHz processor, the GBT analyses typically run under 1 or 2 minutes, whereas using brick elements and ATENA (Cervenka et al. 2013), the analyses can take more than 12 hours if extensive cracking occurs. Furthermore, analytical solutions for elastic shear lag were derived and the unique modal decomposition features of GBT were employed to extract valuable information concerning the effect of shear lag in both the linear and non-linear stages, up to collapse.

For illustrative purposes, Fig. 9 shows the results obtained for a wide flange steel-concrete beam, simply supported and subjected to a uniformly distributed vertical load acting in the web plane. A comparison with the brick model showed that the load-displacement curve is very accurately captured up to 0.2 m, beyond which a shear connection failure is reported in the brick model. Nevertheless, the difference between the maximum loads of the two models is below 1 %.

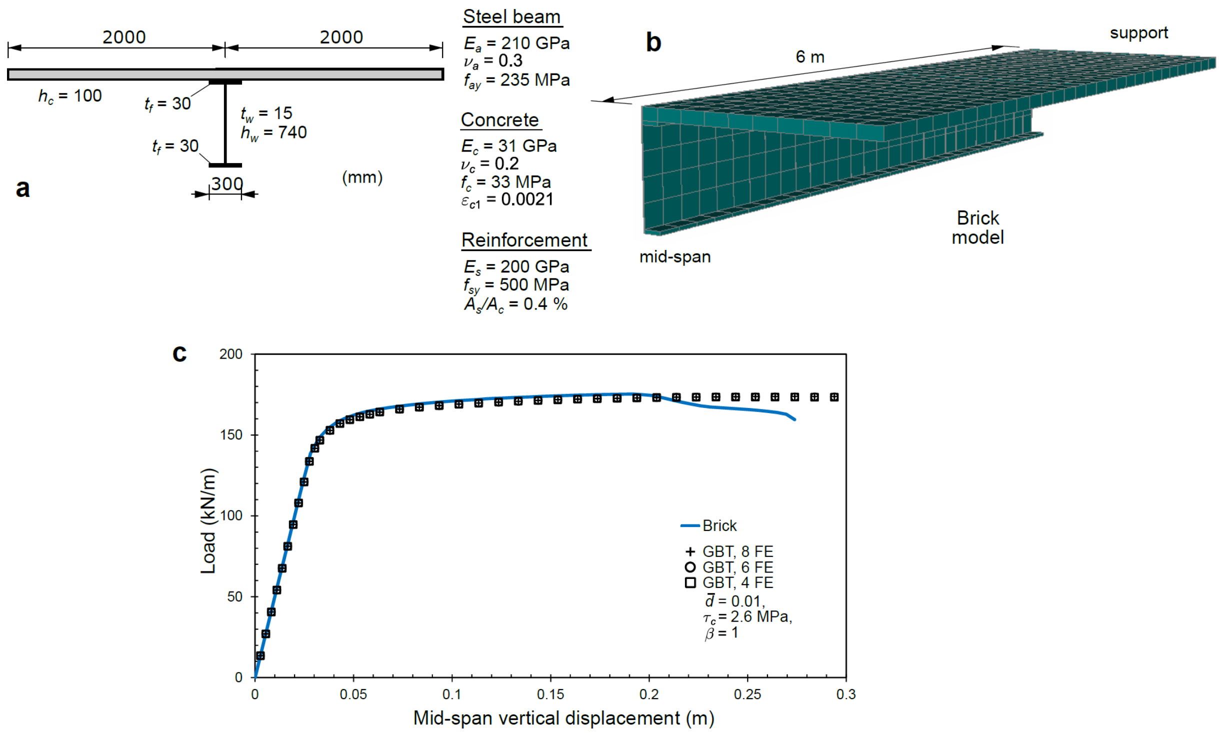


Figure 9: Simply supported steel-concrete beam: (a) geometry and material parameters, (b) brick model and (c) load-displacement plot.

Finally, in Henriques et al. (2016), a GBT-based beam finite element was developed for calculating, accurately and efficiently, elastic bifurcation loads of steel-concrete composite beams. The element includes only a few deformation modes (up to 8 for the pre-buckling analysis and 16 for the buckling analysis) and accounts for shear lag, concrete cracking, shear connection stiffness, cross-section distortion, local (plate-type) deformation in the web and discrete changes of the cross-section. The construction sequence and creep can also be taken into consideration, although in a simplified manner. The element enables attributing independent values to the various bending and membrane stiffness terms of the concrete slab, a feature that allows, for instance, complying with the principles of the U-frame model prescribed in Eurocode 4 (which is not easily achieved using shell elements). The results obtained showed that buckling generally involves a combination of distortion and local-plate deformation modes, even in beams with stocky webs. Fig. 10 displays the critical moments, the associated buckling mode shapes and modal participations, for simply supported beams with two web heights and subjected to uniform negative moment.

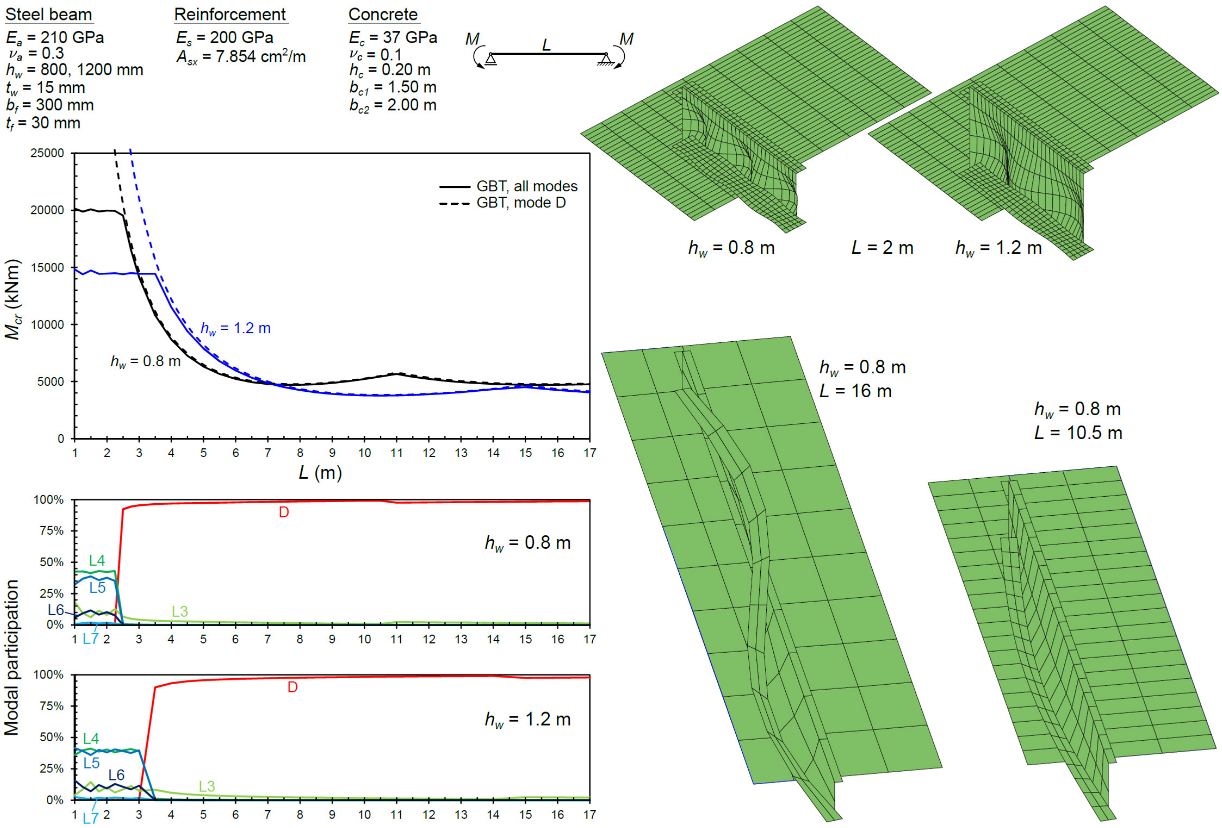
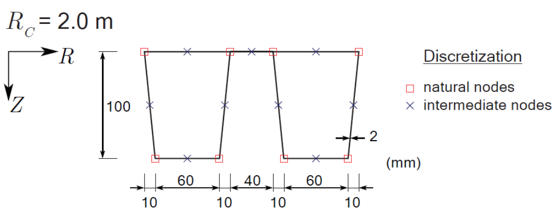


Figure 10: Buckling of simply supported steel-concrete beams subjected to uniform moment.

2.5 Curved bars

The application of GBT to naturally curved bars is rather recent (Peres et al. 2016, 2018). The first paper addressed the development and finite element implementation of a first-order formulation for members with constant bending curvature. Although more complex that for the prismatic case, the standard GBT kinematic assumptions (namely Kirchhoff's, Vlasov's and the null transverse membrane extension assumptions) were incorporated, as they constitute essential ingredients for the general efficiency of the resulting finite element. Furthermore, the equilibrium equations were expressed in terms of GBT modal matrices (the standard approach) as well as stress resultants, which enabled recovering the classic Winkler (in-plane case) and Vlasov (out-of-plane case) equilibrium equations and relations. Although this formulation is capable of handling all types of deformation modes, their systematic determination for complex cross sections was only addressed in the second paper, since the so-called "natural Vlasov modes" (those complying with the Vlasov assumption) involves a constraint that is significantly more complex for curved bars. For this reason, only cross-sections without Vlasov distortional modes were addressed in the first paper. The second paper thus establishes a systematic procedure for the determination of the deformation modes for arbitrary flat-walled cross-sections. The procedure extends the concepts already devised for the prismatic case, namely the modes are hierarchical and subdivided according to specific kinematic constraints that render the GBT analyses quite efficient. Several examples, involving complex local-global deformation, were solved using a standard GBT-based finite element and it was shown that accurate results are obtained with only a few deformation modes and finite elements. It was also observed that, in contrast with prismatic members, the influence length of the local and distortional modes can be quite large and these modes may play a relevant role in zones located far away from the points of load application.

Fig. 11 shows the results obtained for a 90º cantilever beam having the cross-section of Fig. 1, subjected to an eccentric vertical force applied at the free end. The table provides the vertical displacement of the tip obtained with a shell model and 50 GBT finite elements (using several mode combinations). It is noted that the rigid-body modes alone do not provide accurate results and that the Vlasov distortional modes are essential to achieve accuracy. A small improvement is obtained by including the remaining modes.



	tip displacement (m)	
Shell	0.0374231	
GBT, RB modes, 50 FE	0.0268902	(28.1%)
GBT, RB+D modes, 50 FE	0.0369405	(1.3%)
GBT, RB+D+LP modes, 50 FE	0.0370172	(1.1%)
GBT, RB+D+LP+cell shear flow modes, 50 FE	0.0370926	(0.9%)
GBT, RB+D+LP+all shear modes, 50 FE	0.0373968	(0.1%)

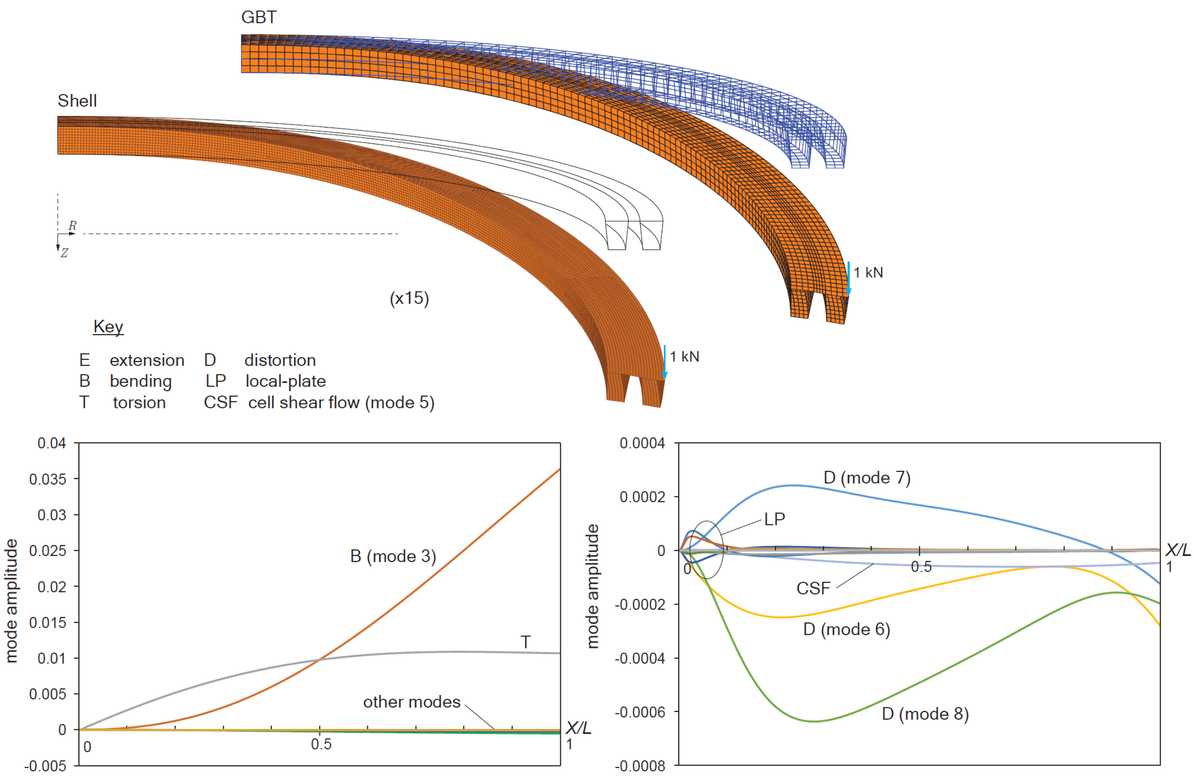


Figure 11: Twin trapezoidal cell section 90º cantilever arch beam subjected to an out-of-plane tip load.

2.6 Cold-formed steel members

GBT has enjoyed widespread application in the field of cold-formed steel member analysis. In Basaglia et al. (2013), the buckling behavior of simply supported lipped channel and zed-section purlins restrained by sheeting, subjected to uplift, was examined using a GBT-based finite element capable of incorporating the effect of discrete and continuous restraints. The variation of the critical buckling moment and mode shape with the purlin length and various restraint stiffness combinations was assessed. Discrete restraints were also investigated for purlins with two typical lengths and the GBT analyses were employed to quantify the minimum translational and rotational stiffness required to ensure an "almost full upper flange restraint". It was shown that the minimum rotational stiffness can be easily attained with commonly used trapezoidal steel sheeting attached to the purlin flange through self-drilling screws. The paper also investigated the case of two-span lipped channel purlins overlapped over the intermediate support, to strengthen the system against the occurrence of local and/or distortional deformations. It was shown that GBT can be easily used to estimate the required strengthening length.

Quite recently, a set of papers examined the mechanics of local-distortional-global interaction in cold formed beams and columns using a geometrically non-linear GBT-based finite element (Martins et al., 2018a-d)

2.7 Other applications

The clear majority of GBT applications concern polygonal cross-sections, since curved geometries render the problem of obtaining the cross-section deformation modes quite complex. Nevertheless, analytical solutions are available for circular tubes and therefore this particular case has been quite explored (Schardt 1989, Schardt 2001, Silvestre 2007, Nedelcu 2011). Elliptical tubes have also been investigated by Silvestre (2008), using sinusoidal functions and series expansions for the curvature radius. In alternative, a polygonal approximation of the (curved) mid-line can be employed, which enables the use of the classic GBT approach, but has the drawback of leading to a very large number of deformation modes, as an accurate geometry description necessarily requires a refined discretization. In (Gonçalves & Camotim 2016) a new GBT cross-section analysis procedure was proposed for the polygonal approximation of curved geometries. The key aspect of this approach is that the geometry is approximated independently of the number of DOFs adopted to obtain the deformation modes. Consequently, it is possible to describe the geometry accurately without increasing dramatically the number of modes. Besides having the major advantage of returning only the most relevant deformation modes, this approach has a noteworthy application to cross-sections with rounded corners, since it allows treating them as polygonal (with sharp corners) from the DOF point of view, while the geometry can still be accurately described.

Bebiano et al. (2017, 2018c) combined GBT with the so-called "exact element method" ("exact" up to machine precision) developed by Eisenberger (1990), to obtain a GBT-based beam element capable of performing buckling and vibration (eigenvalue) analyses of thin-walled members. The power series method for differential equations is used to find the "exact" amplitude functions within the bar, the kinematic boundary conditions are prescribed and the static boundary conditions define the eigenvalue problem. It was shown that the proposed element provides virtually "exact" results while minimizing the DOF number.

3. Geometrically exact beam theories

As already mentioned, GBT offers many advantages in the small-to-moderate displacement range. For large displacements (although small strains), it is necessary to employ more advanced kinematic descriptions. Recently, there has been a growing interest in the so-called "geometrically exact beam theory", pioneered by Reissner (1972) and Simo (1985), which owes its name to the fact that it remains valid independently of the magnitude of the displacements and rotations involved. In its original form, the cross-sections are assumed to remain plane and undeformed, as the kinematic description is based on the position vector of the cross-section axis and the cross-section (finite) rotation, using a rotation tensor, which complicates significantly the formulation. Subsequently, several researchers have included additional deformation modes, namely a torsion-related warping mode (e.g., Simo & Vu-Quoc 1991, Gruttmann et al. 2000, Atluri et al. 2001).

The first paper for thin-walled members with deformable cross-section is (Gonçalves et al., 2010c). In this paper, arbitrary deformation modes are included to model cross-section in-plane and out-of-plane deformation, although Kirchhoff's thin plate assumption is assumed. The deformation modes are written in a co-rotational frame that moves with the cross-section. The examples presented showed that the resulting beam finite element can obtain very accurate load-displacement paths in problems involving complex local-global deformation, even with a small number of finite elements and deformation modes. For illustrative purposes, Fig. 12 shows several deformed configurations of a channel cantilever beam, obtained with only eight geometrically exact beam finite elements. Attention is called to the cross-section distortion appearing near the support, for the higher loading stage.

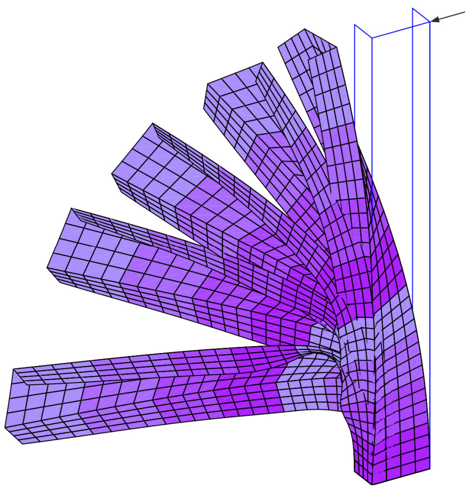


Figure 12: Lateral-torsional buckling of a channel cantilever beam.

With this formulation, an adequate modeling of moderate-to-large cross-section distortion requires the inclusion of linear and quadratic (at least) deformation modes allowing for the transverse (cross-section in-plane) extension of the walls. This requirement is mostly due to a need to describe accurately the relative rotations of the walls, rather than to allow for actual transverse extension (hardly relevant for a broad range of beam-type problems). This led to the development of a formulation with an improved kinematic description, which includes, besides arbitrary deformation modes, finite relative rotations of the walls in the cross-section plane, described by additional rotation tensors (Gonçalves et al. 2011). These “local” or “co-rotational” rotations enable a simple and meaningful geometric description of the cross-section in-plane distortion and, in addition, enable enforcing the transverse inextensibility of the walls. Furthermore, the compound local-global rotations can be used to map the wall local-plate displacements more accurately. The examples presented showed that the formulation enhances considerably the accuracy of the previous one and makes it unnecessary to introduce the linear and quadratic transverse extension deformation modes. The shortcoming of this formulation is that its complexity increases with the number of walls (and thus relative rotations). Fig. 13 shows an example concerning the bending of a channel cantilever, which undergoes severe cross-section flattening.

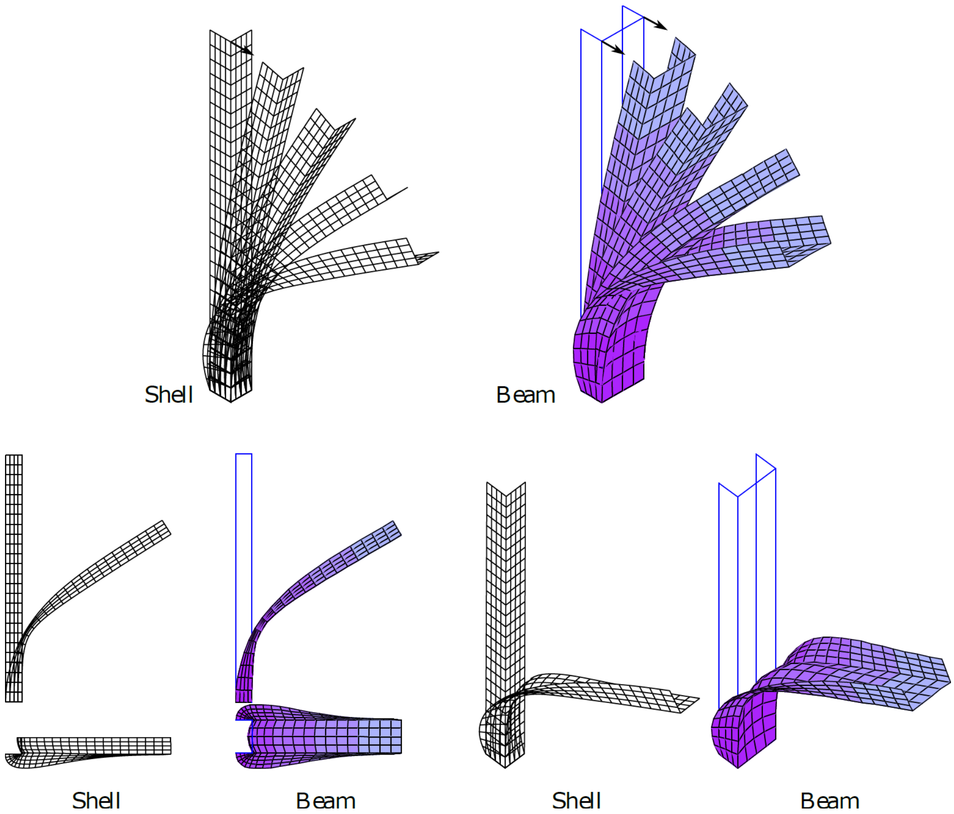


Figure 13: Bending of a channel cantilever beam.

The need to accurately model the relative rotations of the walls inspired the development of a semi-analytic method for obtaining the post-critical behavior of uniformly compressed thin walled members, based on the Koiter approach and the method of separation of variables (Garcea et al. 2017). In this method, the initial post-critical behavior of the column is obtained using a decoupled sinusoidal series solution along the beam axis and a specialized low-cost integration scheme.

In Gonçalves (2016), the computational advantages of combining a shell-like stress resultant elastoplastic law (Ilyushin) with geometrically exact thin-walled beam finite elements were demonstrated. The stress resultant-based material model makes it possible to (i) avoid through-thickness numeric integration (which is mandatory for problems involving torsion) and, together with the enforcement of appropriate kinematic constraints, (ii) set specific stress resultants to zero, leading to particularly simple forms of the return mapping algorithm and the consistent constitutive tangent matrix. The constitutive model was implemented in a two-node beam finite element allowing for torsion-warping and Wagner effects. Fig. 14 concerns an example initially analyzed by Gruttmann et al. (2000). It is observed that the proposed approach yields very accurate results and that the uniform torsion stresses (resultant m_{23}) cannot be considered elastic.

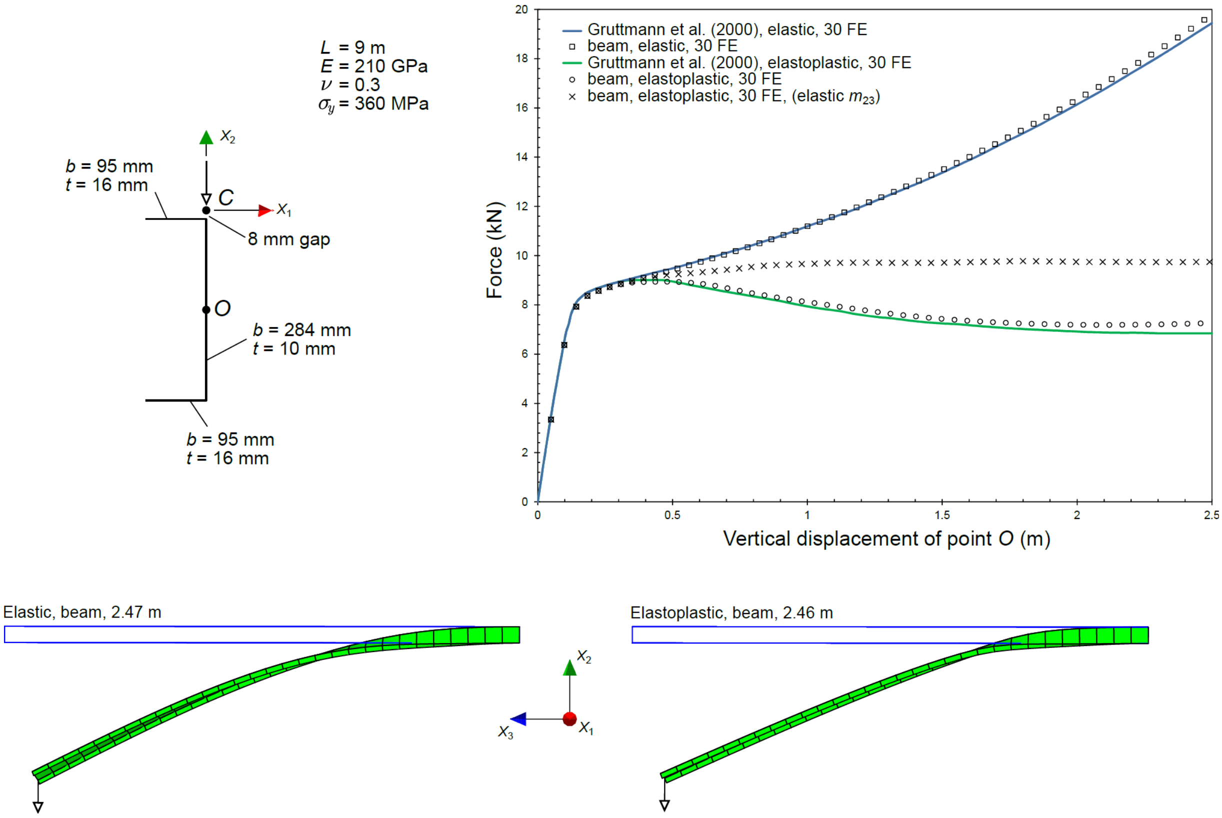


Figure 14: Lateral-torsional buckling of a channel cantilever beam.

In the classic geometrically exact beam theory, the Euler-Bernoulli/Kirchhoff constraints are not employed, as the cross-section translation and rotation are independent. However, these constraints are of major relevance for slender beams, since they eliminate shear locking *a priori* and, in the 2D case, allow using uniaxial material laws and no singularities exist, which is very efficient from a computational point of view. However, on the other hand, the formulation may become quite complex due to the translation/rotation coupling. This type of formulation was first developed in (Weiss 2002a-b, Boyer & Primault 2004), for straight beams with circular cross-section, in which case it is not necessary to keep track of the cross-section in-plane axes.

The author has followed this approach in several papers. As proposed by Boyer & Primault (2004), the cross-section rotation is obtained from the tangent vector of the beam axis, combined with a torsional rotation, using the so-called smallest rotation parametrization. In (Gonçalves 2002), a geometrically exact formulation of this type was proposed for calculating the buckling (bifurcation) loads of thin-walled members with deformable cross-section. The kinematic description employs rotation tensors for the cross-section rotation and the relative wall rotations, besides arbitrary deformation modes complying with Kirchhoff's thin-plate assumption. This enables capturing load height effects associated with coupled cross-section rotation and in-plane distortion. This effect is illustrated in Fig. 15, where a hat section cantilever beam is loaded at the tip, at different heights.

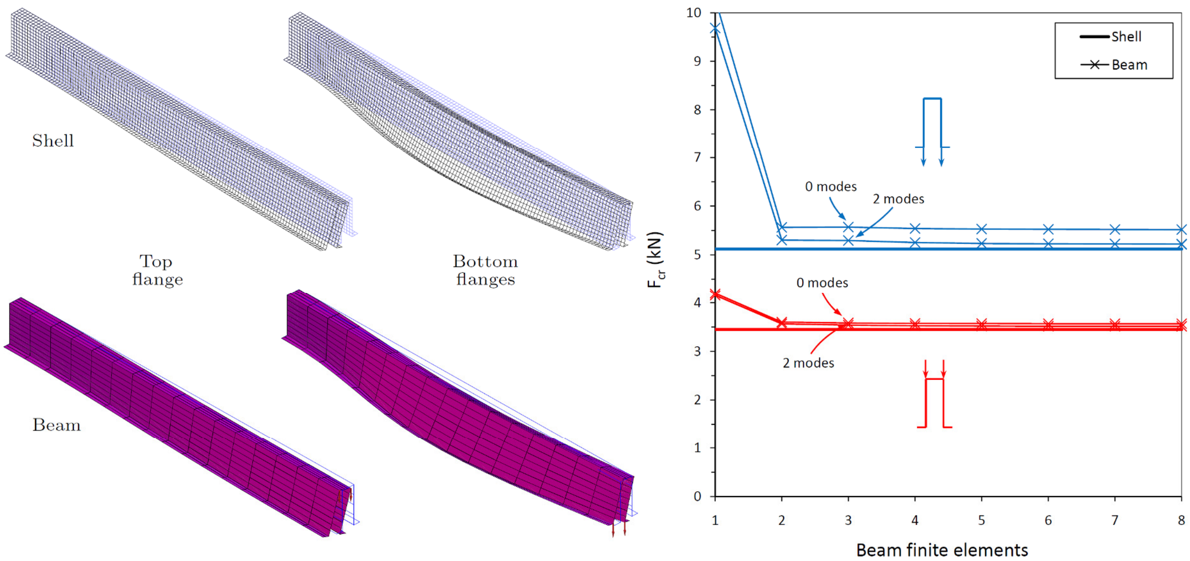


Figure 15: Hat section cantilever beam critical buckling mode shapes and associated loads.

A new geometrically exact Kirchhoff beam model including torsion-warping and Wagner effects was proposed in (Manta & Gonçalves 2016). The numerical examples presented showed that the finite element implementation of the proposed formulation is capable of capturing, very accurately, the spatial behavior of initially straight thin-walled beams, with non-coincident shear centre and centroid, undergoing large displacements and subjected to eccentric forces. A thorough discussion of the particularities of the smallest rotation parametrization was presented and closed-form expressions for the strain measures and their variations were developed. The equilibrium equations and their linearization were fully written in terms of the independent kinematic parameters and can be straightforwardly implemented.

Fig. 16 displays results concerning the large displacement analysis of an I-section cantilever. To trigger lateral displacements, a perturbation force F_2 is applied and kept constant throughout the analysis, while the vertical force F_1 is increased. The small strain version of the proposed element, which is unable to capture the Wagner effect, eventually leads to wrong results. On the other hand, the finite strain version captures the Wagner effect and leads to results that virtually match those obtained with a refined shell finite element model, even with only six equal-length finite elements.

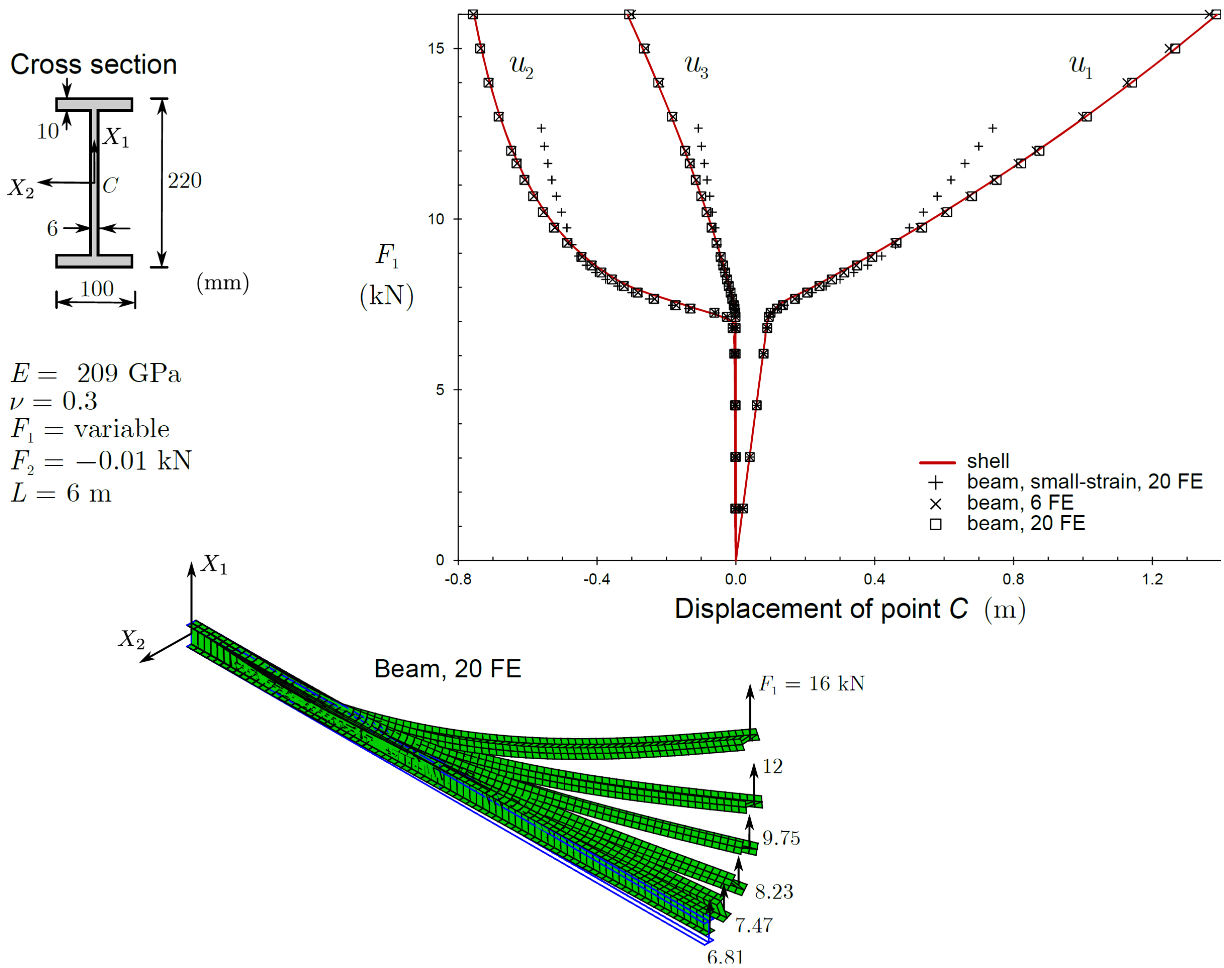


Figure 16: Lateral-torsional buckling of an I-section cantilever beam.

The plane problem was addressed in (Gonçalves & Carvalho 2014, Gonçalves 2018). In the first case, the formulation applies to straight or marginally bent members made of elastic, elastoplastic (steel) or concrete-type material behavior. The resulting finite element proved to be very easy to implement (all relevant expressions are provided in the paper) and extremely accurate, since the validation examples showed that the buckling behavior of steel, reinforced concrete and steel-concrete composite columns may be accurately predicted with just a few finite elements. In the second paper, the formulation was generalized to initially curved members. As in the classic theory of curved bars, a non-linear cross-section stress distribution due to bending is retrieved for small curvature radii. The initial (curved) configuration of each element is obtained from the coordinates of four points along its axis, leading to a particularly easy treatment of complex geometries. Although continuity of the slopes at nodes is not ensured for finite-length elements, smooth and complex curved geometries can be accurately represented with a very small number of elements. At nodes connecting two elements, a single Lagrange multiplier equation is set up to maintain the angles between connecting elements (for n bars connecting at a node, $n-1$ equations are set). A similar approach is used for support conditions involving fixed rotations. As in the previous cases, the element is quite easy to implement, as all relevant expressions are provided in matrix form. Steel plasticity with/without kinematic hardening was implemented, together with concrete cracking/crushing with a mesh-dependency mitigation strategy. The numerical examples presented in the paper demonstrate that the element provides very accurate results in a wide range of cases and that it is very fast, particularly in problems involving concrete non-linearity. For instance, with an Intel Core i7 CPU @ 2.10 GHz processor, the load-displacement path of a partially concrete encased steel I-beam is obtained in less than 5 seconds in a MATLAB (2010) implementation of the proposed element, whereas with a brick model (ATENA), more than 3 hours are required.

Fig. 17 shows a 720º circular beam, fixed at one end, acted by a horizontal load at the free end. The right graph compares the displacement components of the tip for discretizations with 20 and 200 equal-length finite elements. An excellent match is observed and, for the maximum load, the differences in the horizontal displacement are 0.2 %. For the discretization with 20 elements, the analysis was performed in three increments only, whose deformed configurations are shown in the bottom of the figure.

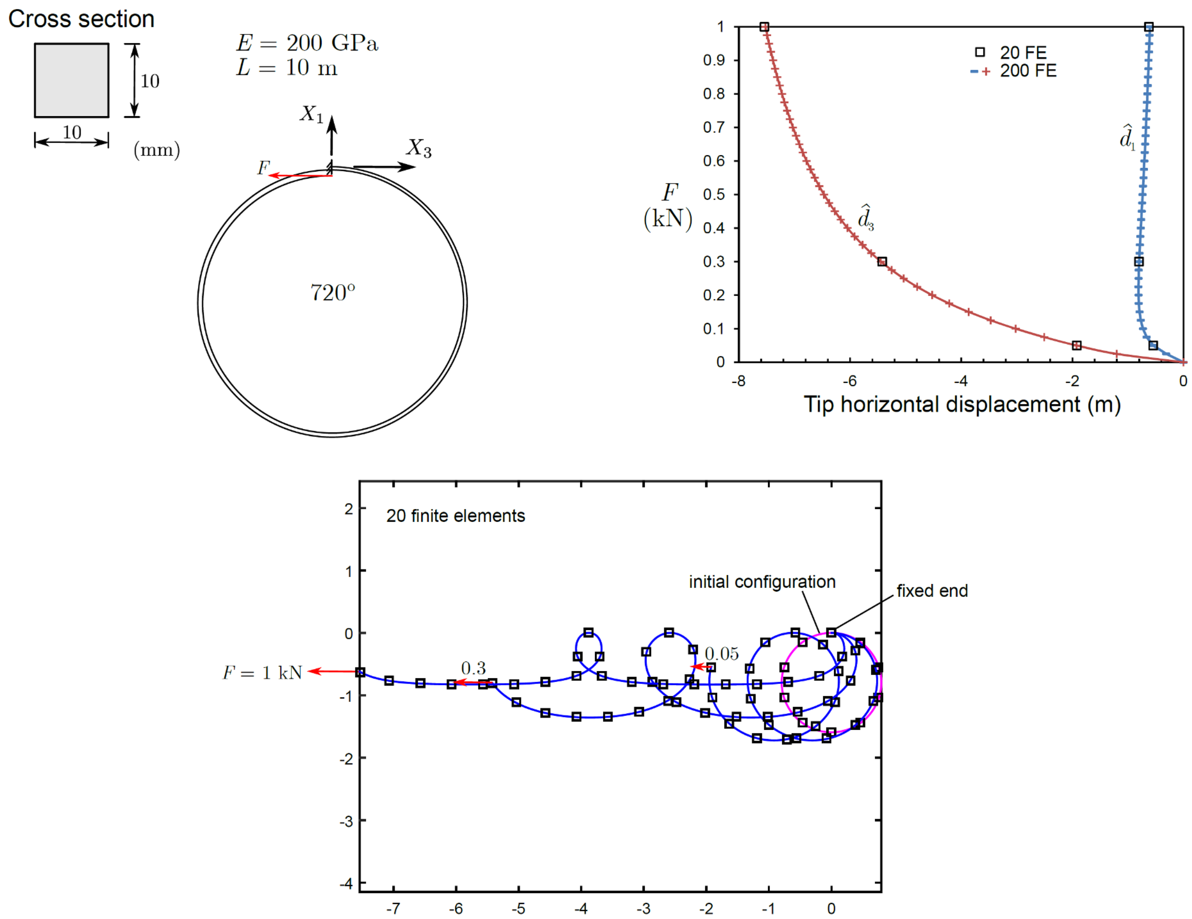


Figure 17: Stretching of a 720º circular beam.

4. Structural design aspects

The buckling (collapse) of partially and fully concrete encased steel I-section uniformly compressed columns according to Eurocode 4 was examined in (Gonçalves & Carvalho 2014), using a geometrically exact Euler-Bernoulli beam finite element, as discussed in Section 3. A parametric study was conducted to assess the differences obtained when the buckling resistance is calculated using either the column buckling curves or the so-called “simplified method”. It was found that the simplified method leads to lower buckling resistances, with differences up to 64% for fully concrete encased columns buckling about the weak axis. From the results of this study, a group of cross-sections was selected for a parametric study using the geometrically exact beam finite element and following the requirements of the Eurocode 4 so-called “general method”. These analyses aimed at assessing the influence of the equivalent member imperfections and the concrete material law parameters. It was concluded that, if design values of the concrete parameters are used and, simultaneously, the exact equivalent imperfection is employed, the buckling resistance generally becomes most close to that provided by the column buckling curves (see Fig. 18). Although further studies are necessary before this approach can be recommended, it should be noted that, for the cases analyzed, the use of mean concrete values, even with the conservative Eurocode 4 imperfections, leads to buckling resistances that in some cases fall well above those obtained from the column buckling curves.

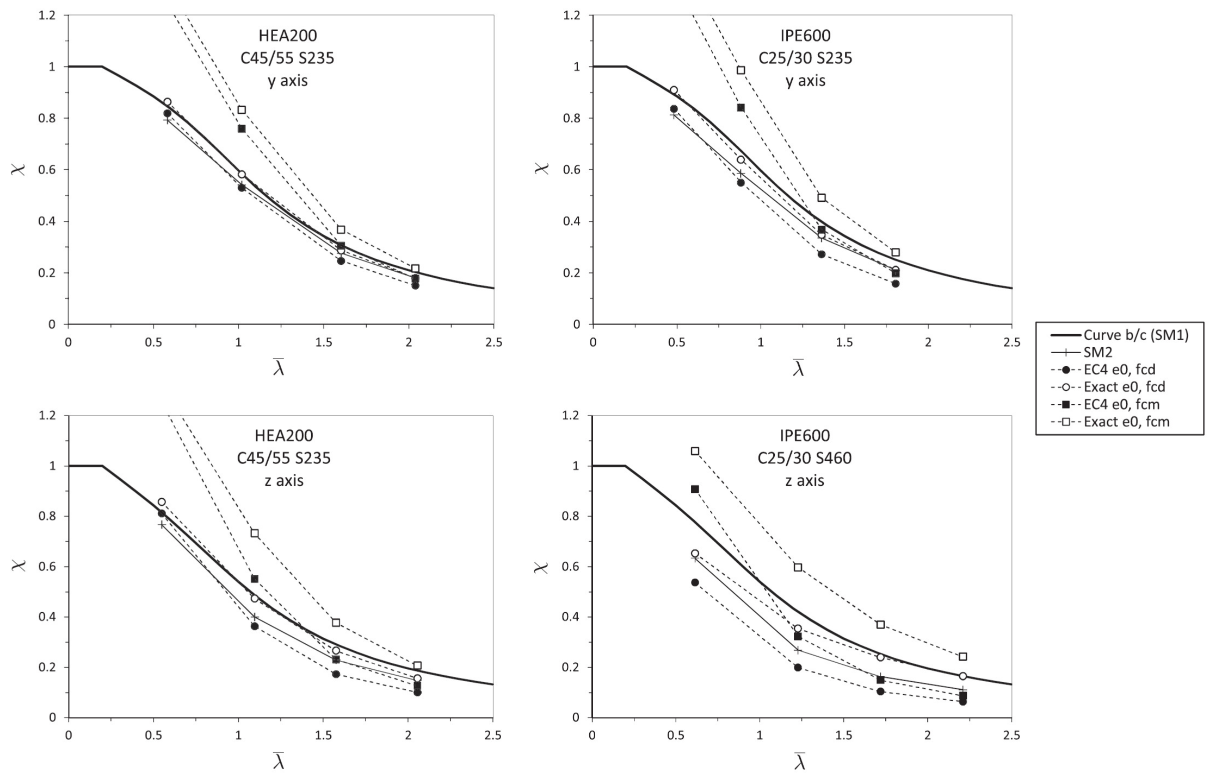


Figure 18: Comparison of the results obtained with the general and simplified methods, for fully encased columns.

In (Gonçalves & Camotim 2017a), a system-based approach for the design of columns integrated in frames, initially proposed in (Gonçalves & Camotim 2005), was extended to continuous beams. This approach can thus be applied to calculate, rationally and efficiently, the buckling resistance of systems formed by members under axial compression or bending. Moreover, it returns a buckling resistance that coincides exactly with that obtained calculating the individual member resistances using the Eurocode 3 buckling curves. This means that the accuracy and reliability of the system-based approach are equivalent to those of the buckling curves (when applied to the individual members). An important aspect of this approach is the fact that it deals with system-based parameters that provide very relevant information concerning the structural behavior of the complete structural system. It was demonstrated that the so-called "system slenderness" plays a crucial role in the characterization of this behavior and in the identification of the member governing the buckling checks. Optimization procedures were also devised. A set of examples were presented, where the resistances provided by the proposed approach were compared with those obtained with non-linear finite element analyses including plasticity, geometric imperfections and residual stresses. The results obtained, such as those shown in Fig. 19, demonstrated the potential of the proposed approach.

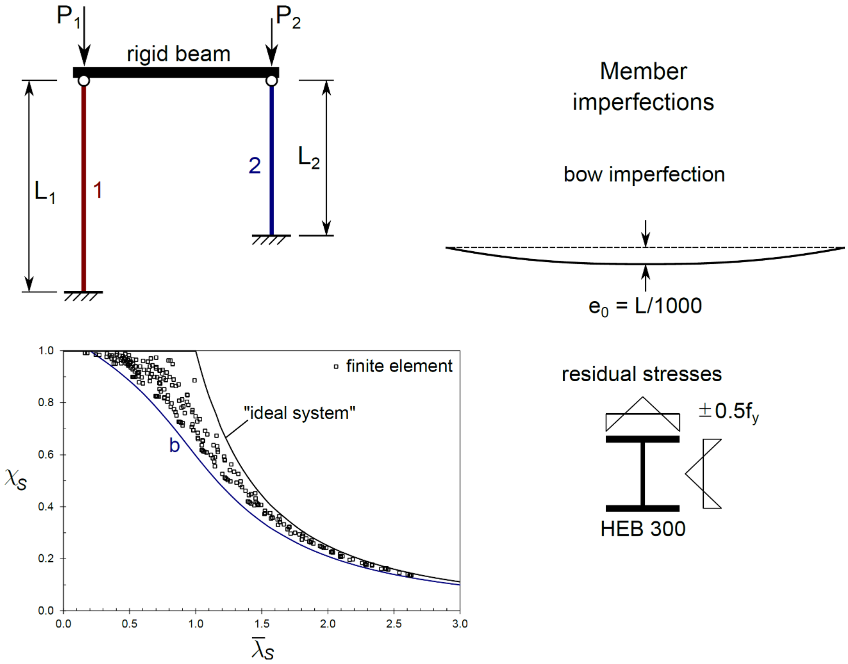


Figure 19: Results of a parametric study concerning a sway frame.

Finally, Gonçalves et al. (2014b) investigated the effect of small-to-moderate shear forces on the plastic moment of compact I-section cantilever beams, for elastic-perfectly plastic materials following the von Mises yield criterion, without hardening. First, the case of rectangular section cantilever beams was examined, subjected to an end shear (see Fig. 20) and axial thrust. It was shown that the classic slip-line field solutions of Green (1954a-b) and Johnson et al. (1974) are in excellent agreement with results obtained with refined plane stress finite element analyses. In particular, as shown by the classic solutions, an increase of the plastic moment is obtained for small-to-moderate shear forces, namely for the so-called "strong support" case. For the so-called "weak support" case, it was observed that the classic solutions correspond to a fixed boundary and that a sliding support is most unfavorable. The weak support case with axial force, not addressed previously, was investigated and it was found that the moment-shear interaction strongly depends on the location of the supported edge.

For I-section cantilever beams subjected to strong axis bending, it was concluded that the strong support with fixed flanges leads to a significant increase of the plastic moment for small-to-moderate shear forces. If, however, the flange mid-lines are axially restrained only, the finite element results become very close to Green's classic solution (Green 1954b). For the weak support, Green's solution yields excellent results up to 35 % of the plastic shear force. It was also shown that transverse stiffeners can easily improve significantly the plastic moment resistance (as in a fixed support).

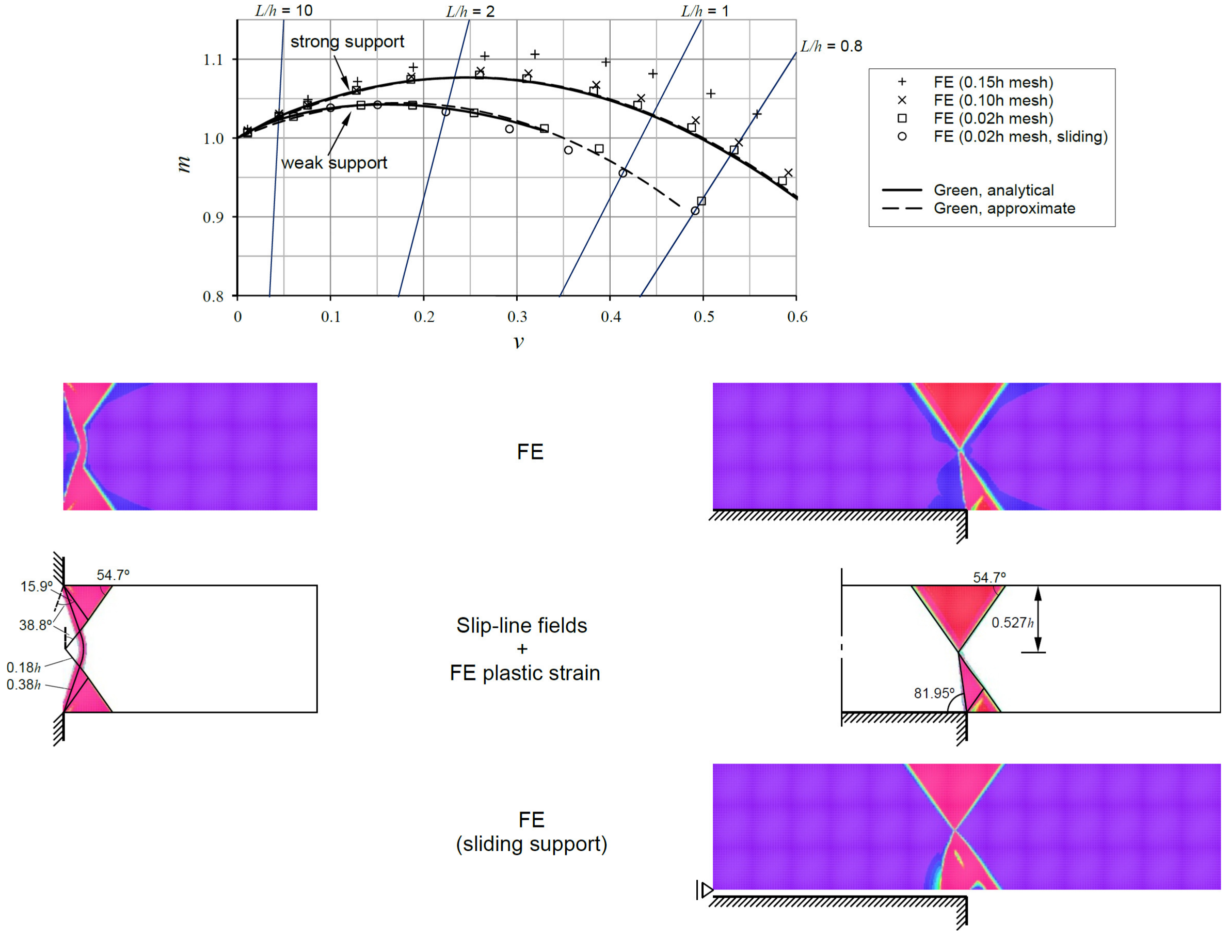


Figure 20: Cantilevers with narrow rectangular cross-section under end shear loading and strong/weak supports: *m*-*v* interaction curves and comparison between FE effective plastic strain distributions near collapse and Green's slip-line field solutions, for *L/h* = 2.

For I-section cantilevers subjected to biaxial bending/shear, a simple interaction equation was proposed, based on the results obtained for rectangular section cantilevers. A comparison with finite element results showed that this equation is quite accurate and generally safe. Moreover, the effect of moderate shear (i) may be discarded for strong supports, (ii) is very small for the so-called "favorable weak support" case and (iii) may be relevant for the so-called "unfavorable weak support" case if vertical shear is present, although for very small cantilever lengths only.

5. Conclusions

This paper reviewed the most relevant research in which the author has been involved, held in the past ten years, in the topic of stability of thin-walled members. Three main work areas were addressed: Generalized Beam Theory, geometrically exact beam formulations and structural design aspects.

Acknowledgments

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