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A modified approach towards estimating the lateral torsional buckling effective length

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Abstract

The lateral torsional buckling (LTB) resistance equations for beams in design specifications require the calculation of an effective length. In lieu of using rigorous analyses, design specifications allow the calculation of an elastic effective length factor (*K*) for beams. The most commonly used method in calculating *K* for LTB resistance of a critical beam span was proposed by Nethercot & Trahair, which makes use of alignment charts for braced columns, while accounting for the restraint provided by its adjoining segments. The authors find that this approach results in values of *K* that are larger or smaller than the true *K* for certain bracing and loading conditions. A *K* estimate that is larger than the true solution adversely effects the formulation of the design LTB curve, wherein a larger beam capacity is ascribed to a given effective length. A *K* estimate that is smaller than the true solution may result in unconservative design solutions. This paper discusses some conditions which are commonly encountered in design and experimental test setups, where unfavourable *K* estimates are made. The examples presented include cases such as multiple equal laterally unbraced lengths under a uniform moment, beams where the critical segment is an end segment, and cases where the critical segment is bounded by unsymmetric restraining conditions at its two braced ends. A modified approach to estimate *K* is presented for such cases, while also considering singly-symmetric cross-sections. The paper is limited to elastic buckling of compact (Class I) I-sections.

1.Introduction

The ultimate strength of a laterally unsupported I-beam with transverse loading is either governed by plastic capacity for short spans, or lateral torsional buckling (LTB) for intermediate and long spans. The elastic LTB capacity of simply-supported doubly-symmetric I-beams subjected to uniform moment derived by Timoshenko & Gere (1961) is

$$M_{cr} = \frac{\pi}{L}\sqrt{EI_yGJ}\sqrt{1+\frac{\pi^2 EC_w}{L^2 GJ}} \tag{1}$$

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where, E is the elastic modulus, J is the St. Venant torsional constant, I_y is the minor axis moment of inertia, C_w is the warping constant and G is the shear modulus.

The determination of LTB capacity depends on the boundary conditions of the critical unbraced length, inelasticity, and the in-plane loading on the critical segment and adjoining segments. Equations in design codes and specifications provide effective length factors for ideal boundary conditions, such as fixed, pinned, etc. Design codes also provide empirical equations to account for inelastic buckling in intermediate length members, along with moment gradient coefficients for various loading conditions. Equation(1) is modified to include these variations to

$$M_{cr} = \frac{C_b \pi}{K L_b} \sqrt{E I_y G J} \sqrt{1 + \frac{\pi^2 E C_w}{(K_b L)^2 G J}} \tag{2}$$

Where C_b is the moment modification factor, K is the elastic effective length factor, and L_b is the unbraced length. However, most practical conditions deal with beams that have intermediate lateral and torsional braces in the form of cross-beams or cross-frames connecting to adjoining parallel beams. Such beams may be simply-supported in-plane, but are continuous in the lateral or out-of-plane direction. The critical unbraced length gets partial torsional fixity from adjoining segments that do not fall under the ideal boundary classification available in design tables. This partial restraint is difficult to ascertain without the aid of computer programs, and most engineers use an elastic LTB effective length factor (K) of 1 for simplicity. However, substantial savings may be achieved by accounting for the appropriate K. Thus, there is a need for a hand-based calculation method to be generally available to engineers. The buckling load of a beam is the LTB capacity of span divided by the maximum end moment in the span

$$P_{cr} = \frac{M_{cr}}{abs \max(M_A, M_B)} \tag{3}$$

where, M_A, M_B are the end moments of the concerned span.

Salvadori (1951) proposed a method to find the critical buckling load of a beam assuming lateral discontinuity at brace points between segments (K=1). The LTB capacity of each segment is calculated using Equation (3) assuming K=1, and the minimum buckling load is taken as the critical buckling load and the span with the critical buckling load is assigned as the critical span. This method gives a good lower bound estimate of the buckling load, but does not take into account the benefits of lateral bending and warping restraints from adjacent segments.

Nethercot and Trahair (1976) proposed a method to calculate the elastic LTB effective length factor based on a column analogy, taking into account lateral bending restraint of adjacent segments. The K factor is calculated by first estimating the fixity at the two ends of the critical segment, and then using the alignment charts or K factor equations in literature for braced columns. This method is briefly outlined below.

The Nethercot and Trahair method

The elastic LTB capacity is calculated using Equation (2).

The moment modification factor, C_b proposed by Salvadori (1955) is

$$C_b = 1.75 + 1.05k + 0.3k^2 \leq 2.56 \tag{4}$$

where k is the ratio of the end moments acting on the segment such that k is less than one.

The elastic effective length factor K is calculated approximately by making use of the similarity between buckling modes of compression member and those of beams under uniform moment. Just as in compression members buckling, stiffness ratios (G_L, G_R) are calculated to estimate K using braced-column alignment charts.

Stiffness ratios given by Equation (5) are the ratios of the member stiffness (α_M) to the restraining member stiffnesses (α_{RR}, α_{RL}) at the right and left ends of the member.

$$G_{L,R} = \frac{\alpha_M}{\alpha_{RL,RR}} \tag{5}$$

The stiffness of the member (α_M) is given by

$$\alpha_M = \frac{2EI_{yM}}{L_M} \tag{6}$$

where L_M is the length of member, and I_{yM} is the minor axis moment of inertia.
The stiffness of the restraining member is given by

$$\alpha_R = \frac{nEI_{yR}}{L_R}\left(1 - \frac{P_{MS}}{P_{RS}}\right) \tag{7}$$

where, L_R is the length of the restraining member, and I_{yR} is the minor axis moment of inertia of the restraining member. The factors P_{MS} and P_{RS} are the buckling load estimates given by Salvadori's method, and n is the coefficient of restraining stiffness given by the far end conditions of the restraining member. The factor n is taken as two when the far end provides a restraint equal to that at the near end, while n is taken as three when the far end is hinged, and n is taken as four when the far end is fixed.

The critical member is identified just as in Salvadori(1951).The restraining stiffness of the adjacent members are calculated based on their far end conditions, and the effective length factor may be calculated using the alignment charts for braced columns. Although C_b is used for accounting for moment gradient effects, the stiffness ratio calculations are principally based on an assumption of equivalent uniform moment concept throughout the unbraced span, which is not always practical.

Although the Nethercot and Trahair method works well for several loading and restraint conditions, the authors observe that there are also several conditions in which the K estimate can be significantly smaller or larger than the true value. A K estimate that is larger than the true value predicts a smaller LTB capacity, and is acceptable from a designer's perspective. However, it adversely affects the formulation of the design LTB curve. An experimentalist may incorrectly ascribe the beam capacity to a larger effective length, thereby shifting the entire design LTB curve to the right while fitting the experimental data. This results in unconservative design, as an engineer uses a larger beam capacity for a given effective length.

The paper presents a number of loading and boundary conditions, and compares the true solutions (obtained via finite element (FE) simulations) and the Nethercot and Trahair (N&T) Method. Further, the paper presents modifications to the method in cases where substantial improvements to the LTB capacity is observed.

2.Modelling

The studies presented in this paper are modelled using finite element software. ABAQUS (2018) and SABRE2 (2016) are extensively used in these studies. Only compact (Class I) I-sections are studied in this paper.

2.1 Material Modelling

The material model for structural steel is used. The studies presented in this paper only deal with elastic buckling. The modulus of elasticity is taken as 29000ksi, and the shear modulus is taken as 0.385E, taking steel Poisson's ratio as 0.3.

2.2 Boundary Conditions

All cases studied in this paper are simply-supported in-plane with a single span. The end supports are laterally and torsionally simply-supported with warping-free restraints (fork boundary conditions). Vlasov kinematics are enforced at the end supports using the equations in Kim (2010) for test simulations with concentrated moments at ends. In this paper, “spans” refer to laterally-unbraced spans, i.e. each beam may have several intermediate torsional restraints, where each of the unbraced segments are referred to as spans.

2.3 Finite Elements and Mesh discretization

The flanges and webs of the I-beams are modelled in ABAQUS using four-node shell elements degenerated from a 3D solid element (S4R shell element). It is a general-purpose reduced-integration shell element that adapts to both thin and thick shell theories. The transverse stiffeners are modelled using the B31 beam element which is compatible with the S4R element.

Following a mesh convergence study, and validation of the models with closed-form solutions presented by Timochenko and Gere (1961), 12 elements are used along the width of the flange, and 20 along the web-depth. The element aspect ratios are maintained close to one.

SABRE2 uses beam elements with seven degrees of freedom (with a warping DOF). Following a mesh convergence study, eight number of elements are used in each unbraced segment. The results obtained thus also provide good correlation with results from ABAQUS.

3.Study

3.1.Restraining effect of spans beyond the immediate adjacent segments

Nethercot and Trahair’s method only takes into account restraining effect of spans immediately adjacent to the critical span. This method has its drawbacks in cases where the number of spans are greater than three. Figure 1 is an example of such a case.

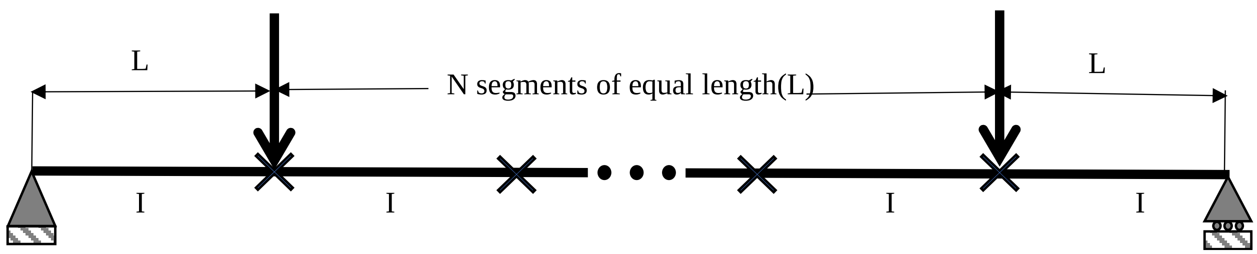

Figure 1: Four point load with equal unbraced segments, with N spans subjected to uniform moment

3.1.1. Four point load test: N spans in uniform moment with equal end span length

In Figure 1, the N&T method identifies the mid segment and its immediate adjacent segments as equally critical, when N is greater than 3. This essentially means that one cannot consider adjacent segments of the same length under the same uniform load to provide any restraint to its adjoining segment. The buckling load of all such segments are equal, and K is taken as one. However, the authors have observed that there can be restraint from segments which are even two or more spans beyond the critical segment. Table 1 tabulates $M_{(FEA)}/M_{(N\&T)}$ for various doubly-symmetric cross-sections and values of N. As seen in Table 1, segments up to 3 spans away from critical have a restraining effect on the critical bucking capacity. However, the effects are increasingly negligible when there are more than two spans at the same buckling capacity and loading as the critical segment. It is further evident from Table 1 that the buckling capacity can be at least 10% larger than that predicted by the N&T method when N = 3. In this case, the second subsequent span is of equal length as the critical segment, but is subjected to a moment gradient (with C_b = 1.75 as per Equation (4)). It is further observed by the authors that this increase in capacity becomes increasingly substantial when the end segment under moment gradient is of smaller lengths than the critical segment. This is discussed subsequently in the paper in more detail.

Table 1: Effect of subsequent spans on the buckling capacity of critical segment (doubly-symmetric sections)

N	3	5	7
$D/t_w = 60$, $D/b_f = 6$, $b_f/t_f = 8$	1.11	1.06	1.03
$D/t_w = 60$, $D/b_f = 4$, $b_f/t_f = 12$	1.12	1.06	1.04
$D/t_w = 60$, $D/b_f = 4$, $b_f/t_f = 8$	1.10	1.04	1.02
$D/t_w = 40$, $D/b_f = 4$, $b_f/t_f = 8$	1.11	1.05	1.03

The shortcoming in the N&T method lies in assuming K=1 in the buckling capacity of the restraining segment, used for estimating the restraining effect as in Equation (8). This assumption is valid when the far end of the restraining member is simply supported, or has zero restraining effect. This is a lower bound buckling capacity of the restraining segment. In the N&T method, the adjacent span's restraining stiffness is given by

$$\alpha_r = n\frac{EI}{L}\left(1-\frac{P_m(K=1)}{P_r(K=1)}\right) \tag{8}$$

If the span adjacent to the far end of the restraining segment has a higher buckling load than the restraining segment, then this segment provides restraint to the restraining segment, thereby increasing its buckling load (i,e. $K_{restraining} < 1$). This larger buckling capacity of the restraining segment results in a larger α_r , and a larger buckling load of the critical span.

The behaviour can be explained by arguing that the segment with a larger buckling load increases the buckling capacity of its immediately adjoining spans having lower buckling capacities. This restraining effect gets carried over several subsequent spans. The critical span can thus benefit from this restraining effect from multiple spans away. Table 1 however shows that this effect diminishes with increasing number of spans, and can be neglected when a segment with larger buckling load is more than two spans away ($N \geq 5$ in Figure 1). This effect can be captured by following these steps:

1. Calculate the buckling capacity of all spans assuming K =1, and appropriate C_b.
2. Identify the segment with the largest buckling load.
3. Use the buckling load of this segment (M_n) to calculate the effective length of its immediate segment in the direction towards the critical segment, and calculate its increased capacity (M_{n-1}). Do this for every segment until the increased capacity of the immediate adjacent span (the "restraining" segment) of the critical segment is found. Use this to calculate the restraining stiffness ratios on the left and right ends (G_L and G_R) of the critical segment.
4. Calculate the effective length of the critical span using the stiffness ratios calculated above.

As noted above, the effects of spans beyond two segments away from critical have negligible impact from a designer's perspective. Thus, in order to make the modified process less cumbersome, it is sufficient to account for the effect of spans up to two segments away. Further, the authors found negligible benefit in making this process iterative, and recommend that the above process be carried out only once. The values of $M_{(FEA)}/M_{n(Pr)}$ for N equal to three in Table 1 by following the above procedure is improved to an average of 1.025 from an average value of 1.11.

3.1.2. Four point load test: 3 spans in uniform moment with varying end span length

Similar to the case in Figure 1, Nethercot and Trahair's method consistently predicts K=1 irrespective of the length of the end segments (mL) in Figure 2.

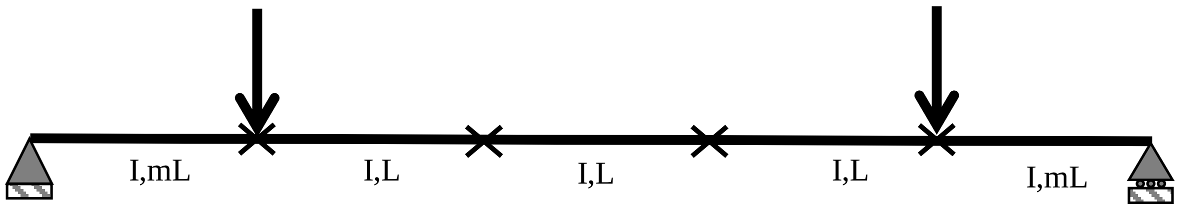

Figure 2: Four point load with 3 equal-length spans subjected to uniform moment, and varying end segment lengths

The case shown in Figure 2 is solved using the proposed method. The graphs in Figure 3 show the relation between the buckling capacity of the critical segment (normalized to the plastic moment capacity as M/M_p) to the elastic effective length. The plots compare the N&T method, the extended method, and ABAQUS results for the cross-section with D/t_w =60, D/b_f =4, and b_f/t_f = 8. With reduction in the length of the end span, the buckling capacity of the critical segment increases as expected. This is also captured by the proposed method. This increase in buckling capacity becomes substantial as the end span length approaches zero as seen in Figure 3. This increased capacity is not captured by the N&T method, while the proposed method is able to capture this effect within ten percent of the ABAQUS results.

Most importantly, Figure 3 clearly illustrates how one may obtain an unconservative LTB curve while using the Nethercot & Trahair method to calculate K for a four-point bending test set-up with only three segments under uniform moment.

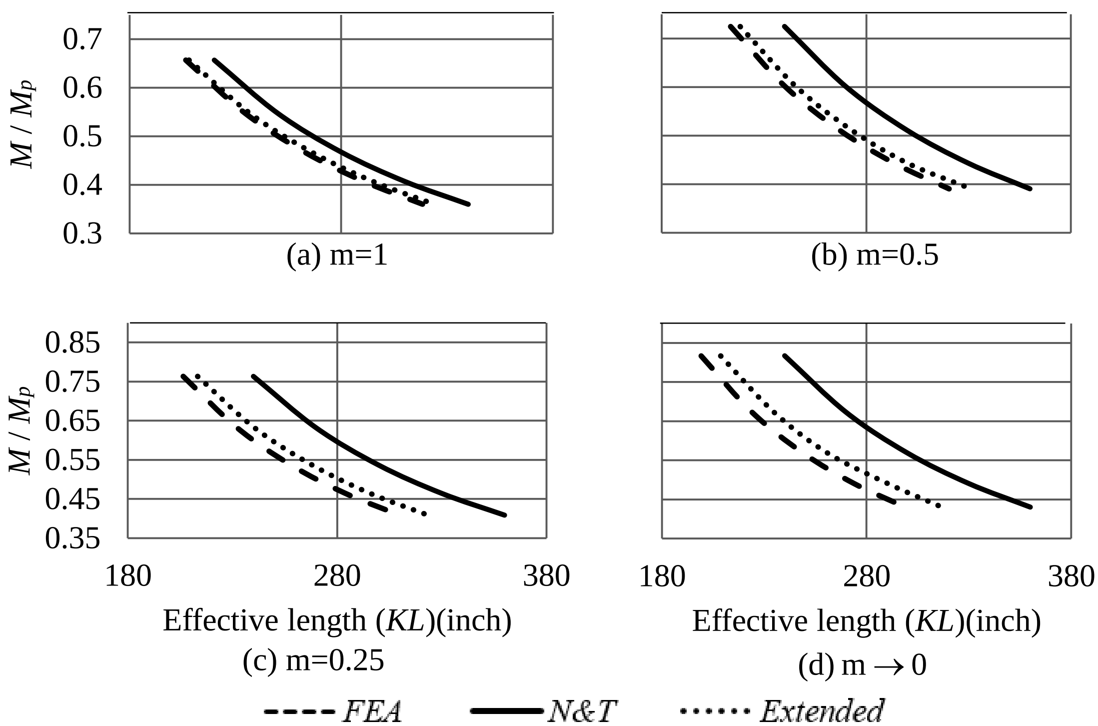

Figure 3: LTB curves of critical segment

3.2. Two-span cases where the critical segment is the end segment

Unlike compression members which have a constant axial stress throughout the length, transversely loaded beams are subjected to varying flexural stresses throughout span. The N&T method treats the loading and boundary conditions independently with moment modification factors (C_b) and effective length factors (K). However, the authors find that this simplified treatment, which neglects the interaction of loading and boundary conditions can predict capacities significantly smaller or larger than the true capacities for various conditions. This is particularly observed for cases with high moment gradients and unsymmetric boundary conditions (or restraint conditions at the two ends of the critical segment). Cases where the end span is critical have unsymmetric boundary conditions, where one end is simply-supported i.e zero restraining end, and the other end is at a brace point, with restraint from adjacent spans. When these end spans are further subjected to high moment gradients throughout the span, this interaction effect is observed. These cases are discussed in this section.

3.2.1. Two span cases (single curvature spans)

Figure 4 shows simply-supported beams with two unbraced segments of different lengths, but same cross-sections. The right segment is designed to be critical; the boundary conditions on the critical segment here is such that it receives restraint from the adjacent span at one end and no restraint on the other end (simple support). Therefore, the boundary condition is referred to as unsymmetric. While the bending moments shown in Figure 4 is purely theoretical, it is chosen here to illustrate the behavior of unsymmetric beams. This concept is then extended to other loading conditions. Figure 4(a) is a case where the maximum moment in the critical segment is adjacent to a restraining segment, while the case in Figure 4(b) has the maximum moment at a zero-restraint end. The length L of the restraining end is varied between one (near-zero length) to 240 inches (equal to the length of the critical segment).

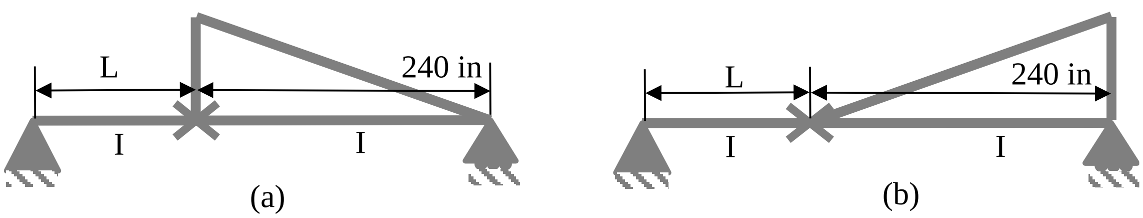

Figure 4: Bending moment diagrams of laterally braced spans

Table 2 shows the difference between FE results from SABRE2 to the estimates from the N&T method for the cases in Figure 4 for D = 30in, b_f = 7.5in, t_w = 0.5in, t_f = 0.9375in. The N&T method predicts the same capacity for both cases shown in Figure 4, while it is evident from the SABRE2 analysis that the N&T predictions can be as much as 36% conservative to 20% unconservative. The N&T method consistently predicts unconservatively for Figure 4(b) and conservatively for Figure 4(a). In Figure 4(a), the location of the maximum bending moment within the critical segment is adjacent to the restraining segment. Thus, the effect of the restraining segment is more pronounced here as opposed to the case in Figure 4(b), where the maximum moment is located at a pin support that offers no restraint. In general, when the larger moment acts closer to the braced point, Nethercot and Trahair's method predicts smaller capacities (conservative), and when the larger moment acts closer to the pin support, Nethercot and Trahair's method predicts larger buckling strengths (unconservative).

Table 2: Comparison of buckling capacities from FE simulations and the N&T method for beams with two unbraced spans in single curvature

L (in)	$M_{(SABRE2)}$ (in-kips) Fig. 4(a)	$M_{(SABRE2)}$ / $M_{(N\&T)}$ Fig. 4(a)	$M_{(N\&T)}$ (in-kips)	$M_{(SABRE2)}$/$M_{(N\&T)}$ Fig. 4(b)	$M_{(SABRE2)}$ (in-kips) Fig. 4 (b)
240	17861	1.16	15329	0.93	14218
120	20636	1.22	16949	0.89	15016
60	23675	1.28	18477	0.86	15803
30	26030	1.33	19634	0.84	16396
15	27460	1.35	20375	0.82	16768
7.5	28232	1.36	20800	0.82	16977
1	28918	1.36	21203	0.81	17172

As the length of the restraining segment (L) is reduced, the unsymmetry in the restraining effect from both ends of the critical span increases. As seen in Table 2, for Figure 4(a) and (b), the N&T predictions become increasingly offset from the true capacities as the length of the restraining segment reduces. From this, it is concluded that an improved procedure to determine K is desirable when high moment gradients are applied to highly unsymmetric boundary conditions.

In the procedure developed for such conditions, referred to as the load-boundary condition (LBC) interaction method, the authors define a term "Distribution Factor", which accounts for the moment gradient within the critical segment, while recognizing the difference in behavior between the braced point being at the location of the maximum or minimum bending moment.

For linearly varying moments, the unsymmetricity of moments can be captured by discretizing moments as given by Equations (9) and (10).

$$M_{far} = 0.75M_L + 0.25M_0 \tag{9}$$

$$M_{near} = 0.25M_L + 0.75M_0 \tag{10}$$

where, M_L is the moment at length L from the braced point (i.e moment at the pin support), and M_0 is the moment at the brace point. The change in moment from the near end to the far end from the concerned brace is captured by defining a Distribution Factor (DF) in Equation (11)

$$0.33 \le (DF) = \frac{M_{far}}{M_{near}} \le 3 \tag{11}$$

A distribution factor smaller than one indicates that the larger moment is at the restraining end (brace point), while a DF greater than one indicates that the larger moment is at the simply-supported end.

3.2.1.1. Conservative prediction cases (DF<1, when maximum moment acts at the brace point)

Nethercot and Trahair's method predicts conservatively for cases where the larger moment acts at the brace point. The following steps detail a modified procedure developed by the authors for these cases, which significantly improve the estimate of K, and the buckling capacities.

1. Calculate the stiffness ratio as defined in Nethercot and Trahair's method, say *G(N&T)*.
2. Calculate the Distribution Factor (DF) using Equation (11)
3. Calculate β using Equation (12)
4. Calculate K using Equation (15) (Dumonteil 1992)

The stiffness ratio calculated using Nethercot and Trahair's method is modified to define a new variable β as in the equation (12)

$$\beta_A = aG_A(N\&T) - b \tag{12}$$

$$a = \begin{cases} \min(1, 2\times DF) & G(\mathrm{N\&T}) < 1 \\ 1 & G(\mathrm{N\&T}) > 1 \end{cases} \tag{13}$$

$$b = \frac{0.06}{DF} \tag{14}$$

$$K_{new} = \frac{3\beta_A\beta_B + 1.4(\beta_A + \beta_B) + 0.64}{3\beta_A\beta_B + 2(\beta_A + \beta_B) + 1.28} \tag{15}$$

The subscripts A and B on β and G denote the values at the two ends of the critical segment. Figure 5 shows the improvement of the proposed method over Nethercot and Trahair's method in calculating K for several loading conditions with end-span critical and smaller moment at the brace point.

The plots in Figure 5 show the predictions of K for different ratios of restraining segment lengths to the critical segment lengths. The following points may be gleaned from these results.

1. The proposed modified method consistently predicts better effective length factors when compared to the conventional Nethorcot and Trahair's method.
2. The N&T method is most conservative for the unloaded restraining segment (Figure 5(a)), with improving accuracy as the bending moment in the beam approaches a uniform bending moment (Figure 5(c)).
3. The N&T method is most conservative at shorter lengths of the restraining segments, as opposed to cases with larger lengths of restraining segments. This is due to the increased unsymmetry in restraints at two ends and the warping restraint provided by shorter segments, but is not accounted for in the N&T method.
4. The proposed method performs consistently well for all loading conditions and for all lengths of the restraining segments. The coefficient of variation (COV) of K_{FEA}/K_{Pr} across different lengths is 0.01 as opposed to a COV = 0.12 for $K_{FEA}/K_{N\&T}$.

3.2.1.2. Unconservative cases (DF > 1, when the maximum moment acts at the pin support)

Nethercot and Trahair's method predicts greater buckling strengths for cases where the larger moment acts at the simply-supported end. This study proposes modifications to the Nethercot and Trahair's method to predict better effective length factors. The proposed steps here are essentially the same as those for the case with the maximum moment in the critical span at the brace location. The difference is only in the expressions for the coefficients a and b in Equation (12) for β.

$$a = \max(\frac{DF}{2}, 1) \tag{16}$$

$$b = \frac{-DF}{10} \tag{17}$$

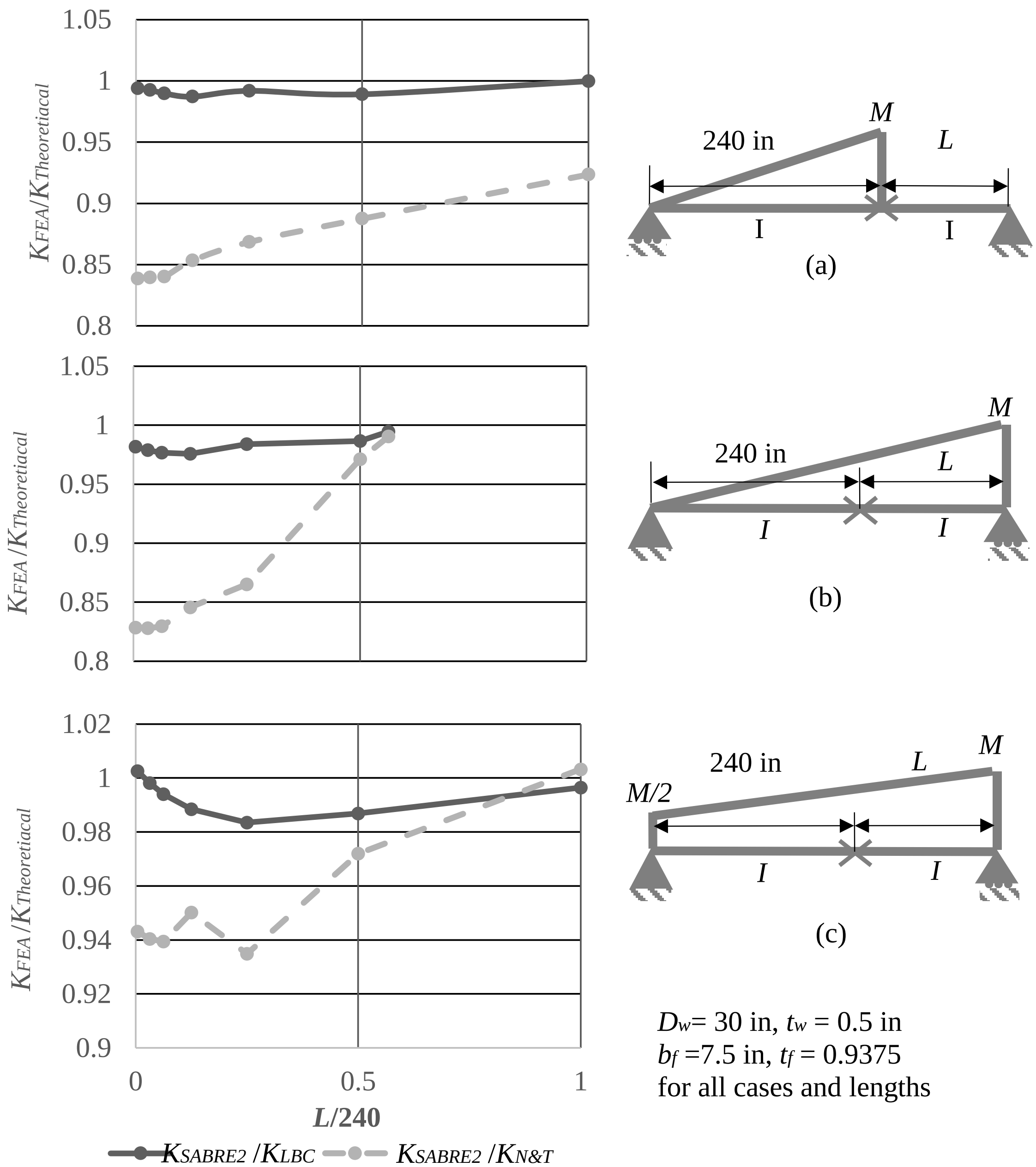

Figure 5: Effective length factor estimates of N&T and proposed methods when maximum moment is at brace point

Figure 6 shows the improvement of the proposed method over Nethercot and Trahair's method in calculating K for several loading conditions with end-span critical and larger moment at the brace point. The following points may be gleaned from these results.

1. The proposed modified method consistently predicts better effective length factors when compared to the conventional Nethorcot and Trahair's method. The proposed method predicts effective length factors conservatively within 3%.
2. The N&T method is most unconservative for the unloaded restraining segment (Figure 6(a)), with improving accuracy as the bending moment in the beam approaches a uniform bending moment (Figure 6(c)).

3. The N&T method is most unconservative at shorter lengths of the restraining segments, as opposed to cases with larger lengths of restraining segments.
4. The proposed method performs consistently well for all loading conditions and for all lengths of the restraining segments. The coefficient of variation (COV) of K_{FEA}/K_{Pr} across different lengths is 0.01 as opposed to a COV = 0.07 for $K_{FEA}/K_{N\&T}$.

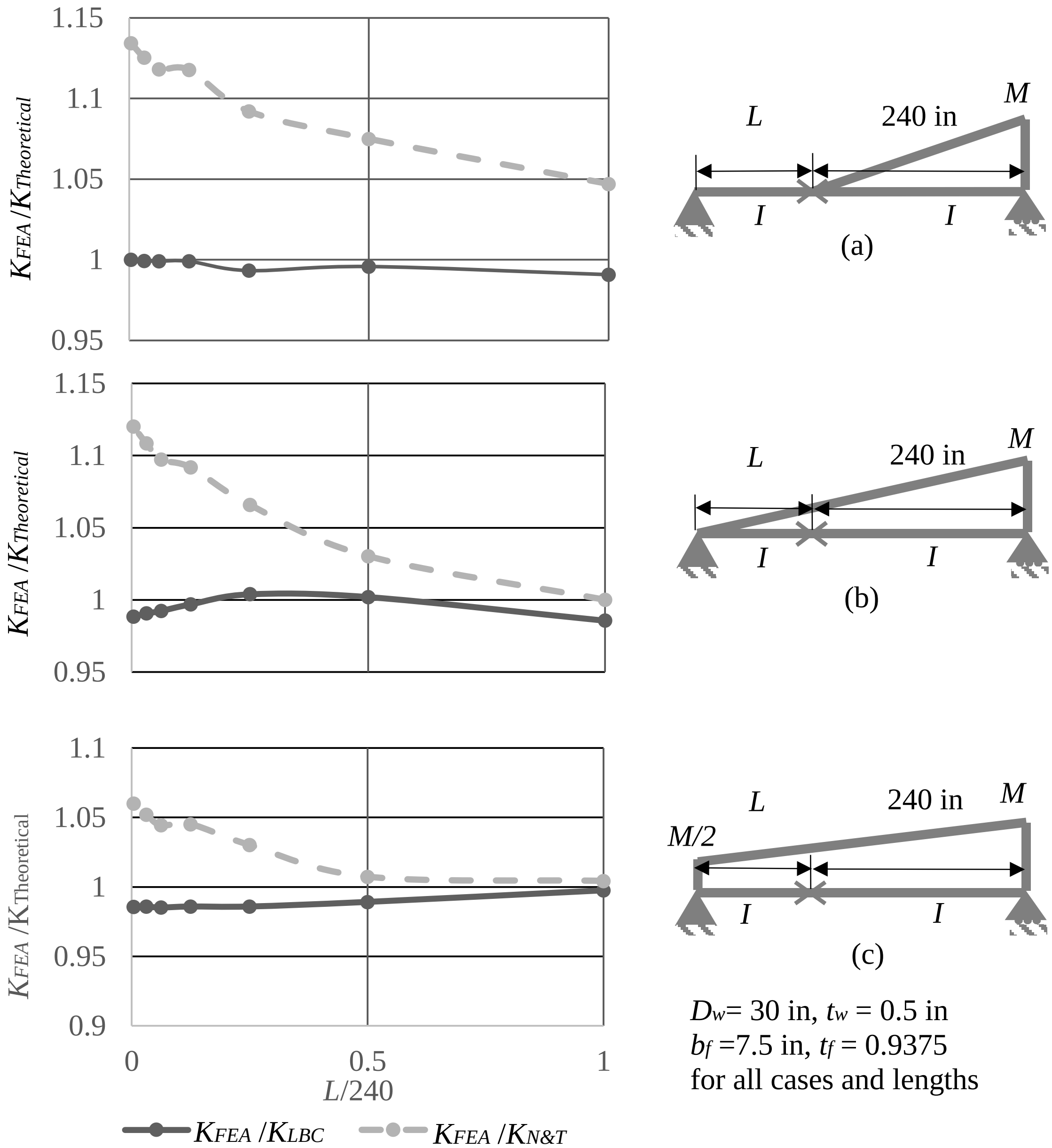

Figure 6: Effective length factor estimates of N&T and proposed methods when maximum moment is at pin location

3.2.2 Two-span cases subjected to reverse curvature

The restraining segment typically loaded to a certain extent, loses its stiffness and its restraining effect with it. To account for this loss of stiffness, a lower bound modification factor $\left(1-\frac{P_m}{P_r}\right)$ is multiplied with the assumed stiffness.

$$\alpha_r = n\frac{EI}{L}\left(1-\frac{P_m}{P_r}\right) \tag{18}$$

In Equation (18), P_m and P_r are the buckling loads of the critical segment and restraining segments respectively. The modification factor in Equation (18) holds valid when the beam is in single curvature, i.e. the compression flange is also restrained by a flange in compression. Conversely, when the beam is subjected to reverse curvature, the compression flange in the critical segment is restrained by a flange in tension. In such a scenario, the modification factor for the loss of stiffness is not valid. In reality, an increased stabilizing effect is observed when the restraining flange is in tension instead of compression. To address this issue, a lower bound estimate assuming zero reduction in stiffness would predict better results.

$$\alpha_r = n\frac{EI}{L} \tag{19}$$

Figure 7 shows the improvement of the proposed method over Nethercot and Trahair's method in calculating K for two reverse curvature cases. The 240 inch span is the critical segment. Figure 7(a) shows a case with equal moments at the two ends of the beam (with the zero bending moment at midspan), while the case in Figure 7(b) has a concentrated moment at brace point. The brace point is at a distance L from the left support, and does not necessarily coincide with the midspan.

The following points may be gleaned from the results shown in Figure 7.

1. The proposed method in general predicts better results than the N&T method for beams subjected to reverse curvature. In Figure 7(a), the coefficient of variation (COV) of K_{FEA}/K_{Pr} across different lengths is 0.06 as opposed to a COV = 0.23 for $K_{FEA}/K_{N\&T}$. In Figure 7(b), while the COV of K_{FEA}/K_{Pr} across different lengths is comparable to the COV of $K_{FEA}/K_{N\&T}$, the predictions via the proposed method are vastly improved (by at least 10%).
2. The proposed method works better for the loading condition in Figure 7(a), when compared to the loading in Figure 7(b). Both the proposed and N&T methods are conservative in Figure 7(b). This may be rationalized by noting that the restraining flange in Figure 7(b) is in tension at the brace location, and has a stabilizing effect that is larger than an unloaded segment. This is conservatively neglected in Equation (19). The N&T method uses Equation (18), which makes it even more conservative.
3. The prediction of the proposed and N&T methods changes from unconservative to conservative in Figure 7(a) because of the location of the intermediate brace relative to the location of zero bending moment. It is unconservative when the critical segment is in reverse curvature, and conservative when the critical segment is in single curvature. However, the proposed method is less than 5% unconservative as compared to 15% by the N&T method.

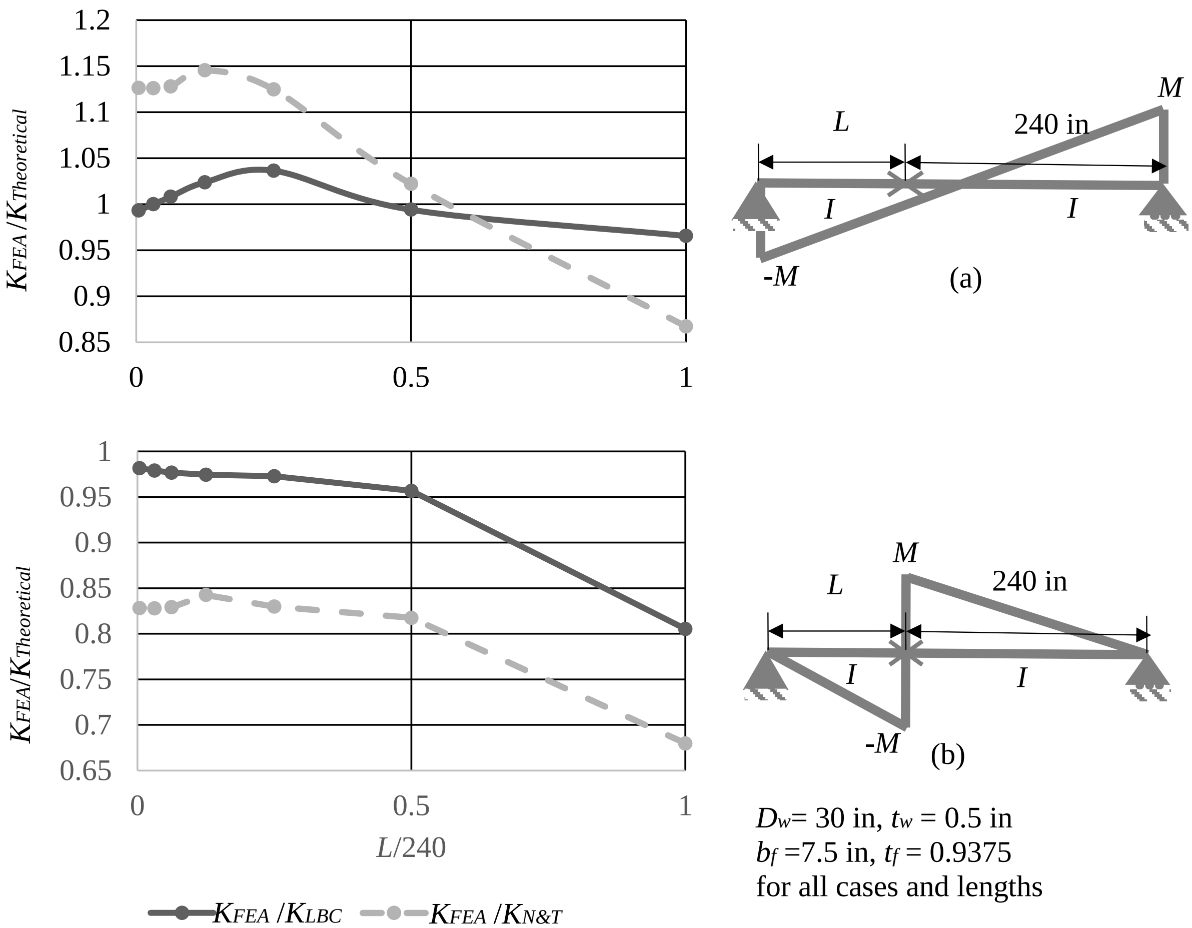

Figure 7: Effective length factor accuracy comparison of N&T and Proposed method

3.3.Three-span cases where the critical segment is the end segment (Three-point bending): Incorrect identification of critical segment

The loading conditions discussed in this Section deal with cases where the current N&T method tends to unconservatively predict the buckling capacities due to an incorrect identification of the critical segment. The two important steps of the N&T method are to identify the most critical segment, and to estimate the restraining effect from its adjacent spans.

Figure 8 shows a three-point loading scenario. It is evident that for cases where L < 240 inches, and assuming K = 1 (as required by the N&T method), the mid-segment is identified as the most critical segment. This is due to the more severe loading in the mid-segment as compared to the two end segments. However, it is observed that the right end-segment is in fact the most critical segment. This is because of the pin support at one of its ends. The mid-segment obtains sufficient restraint at both its ends, which increases its buckling capacity despite its higher load. This raises the question on whether simply using K = 1 on each segment is a valid method of identifying the critical segment.

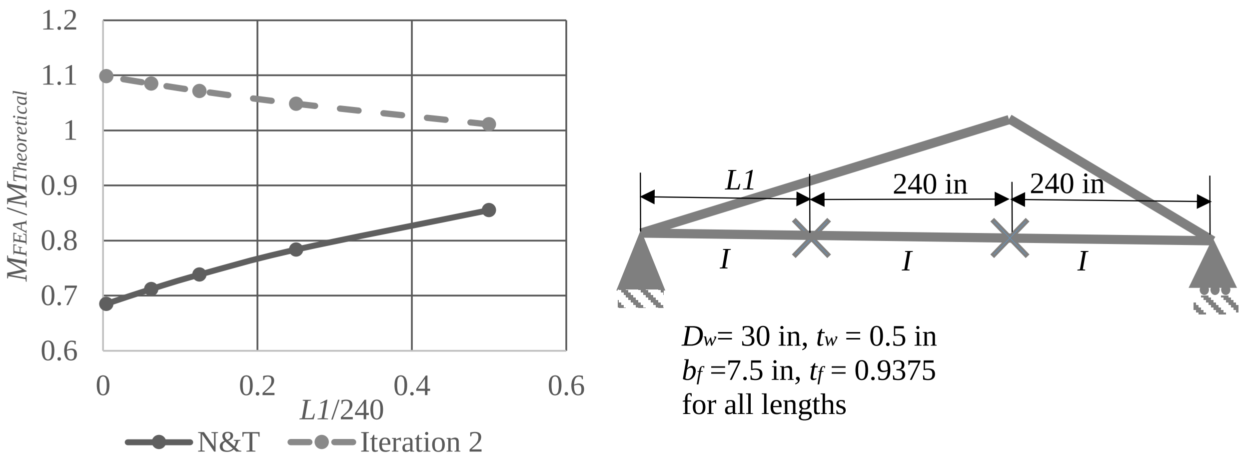

Figure 8: Comparison of the proposed iteration method to the N&T method when critical segment is re-evaluated after first iteration

In order to address this shortcoming, the authors recommend that the N&T method may still be used, but with a caveat that may require an iterative process. If one uses the N&T method, and computes the buckling capacity of the critical segment (incorrectly identified as the mid-segment), it is found that the final buckling capacity is larger than the buckling capacity of the right end-segment at the end of the N&T process. If and when this occurs, one may repeat the N&T method a second time by using the buckling loads calculated in the first pass. Obviously, the right end-segment should be considered as the critical segment in the second iteration, owing to its smaller buckling capacity. This iterative process consistently predicts capacities which are conservative within 10% as seen in Figure 8. It is important to note that the N&T method predicts capacities that can be up to 30% unconservative in such cases.

3.4. Three-span cases where the critical segment is the middle segment (Three-point bending): Incorrect estimation of restraining effect

The previous section discussed the possibility of incorrectly identifying the critical segment. This section discusses cases where the N&T method can be significantly unconservative, and may require an iterative process due to significant over-estimation of the restraining effect of the adjacent spans. The restraining effect of the adjacent spans is given by

$$\alpha_{r0} = n\frac{EI}{L}\left(1 - \frac{P_m(K=1)}{P_r(K=1)}\right) \tag{20}$$

The critical buckling capacity calculated using the N&T method is say P_{m1}. If $P_{m1} >> P_m(K=1) \Rightarrow \alpha_{r1} \ll \alpha_{r0}$ (α_{r1} is the restraining stiffness calculated using the new buckling load P_{m1}, α_{r0} is the restraining stiffness calculated initially), i.e. the restraining effect reduces significantly from that estimated in the first pass. This effect is particularly significant when the buckling capacity of the restraining member and the change in the critical member buckling load (P_{m1}- P_{m0}) are comparable. This problem demands an iterative process as suggested in Section 3.3 to predict better results. Such a case is shown in Figure 9 comparing the iteration method to the single-pass N&T method.

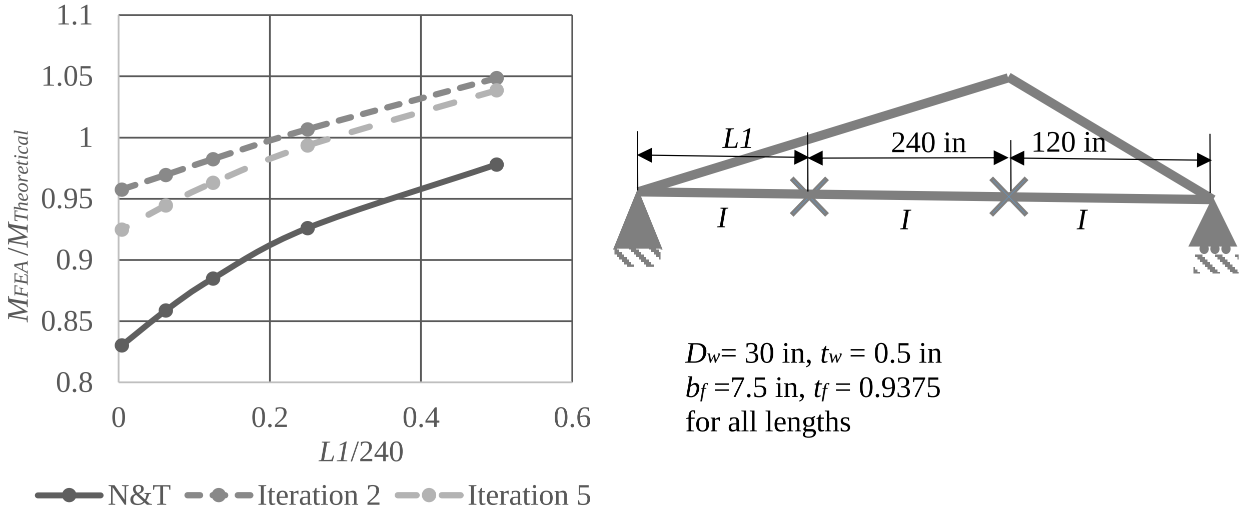

Figure 9: Comparison of the proposed iteration method to the N&T method when the restraining effect is re-evaluated after the first iteration

As seen in Figure 9, the iteration method predicts the buckling load with greater accuracy as the change in buckling capacity increases. It is also observed that Iteration 5 predicts very similar buckling loads to Iteration 2 (within 3%). It is sufficient to repeat the N&T method just once when the first pass satisfies the condition stated above.

3.5 Singly-symmetric sections

The Nethercot and Trahair method, and all the modifications proposed in this paper for doubly-symmetric sections can be extended to include singly-symmetric I-sections. This paper considers cross-sections with the larger flange in tension (considered more critical for LTB). The LTB buckling capacity for singly-symmetric cross-sections given by Timoshenko & Gere (1961) is

$$M_{cr} = \frac{C_b \pi}{KL}\sqrt{EI_y GJ}\left[\frac{\pi \beta_x}{2KL}\sqrt{\frac{EI_y}{GJ}} \pm \sqrt{1+\frac{\pi^2 \beta_x^2}{4K^2 L^2}\left(\frac{EI_y}{GJ}\right)}\right] \tag{21}$$

The stiffness ratios are calculated as the ratio of member stiffness to restraining member stiffness ($G_L = \frac{\alpha_M}{\alpha_{RL}}$ and $G_R = \frac{\alpha_M}{\alpha_{RR}}$), where α_M is the member stiffness for singly-symmetric sections, and given by Equation (22)

$$\alpha_M = \frac{2(b_{fc}t_{fc} + h_c t_w / 6) r_t^2}{L_{b,critical}} \tag{22}$$

and the restraining member stiffnesses for singly-symmetric sections is given by Equation (23)

$$\alpha_{RL,RR} = \frac{n(b_{fc}t_{fc} + h_c t_w / 6) r_t^2}{L_{bL,bR}}\left(1 - \frac{P_{MS}}{P_{RL,RR}}\right) \tag{23}$$

The term ($b_{fc}t_{fc}+h_c t_w/6$) is derived for singly-symmetric sections by taking the compression flange and one-sixth the web depth in compression as equivalent compression columns given in White and Jung (2003) and White (2008). In the above equations, r_t is effective radius of gyration for LTB, $L_{bL,bR}$ is the length of restraining segment on the left and right respectively, $L_{b.critical}$ is the length of the critical segment, b_{fc} is the width of the compression flange, t_{fc} is the thickness of the compression flange, h_c is the depth of the web in compression, t_w is the thickness of the web and β_x is a cross sectional property for singly-symmetric sections.

The case in

Figure 10 is solved using the extended method to account for restraining effect of spans beyond adjacent span to critical span (detailed in Section 3.1) and the loading-boundary condition (LBC) interaction method (detailed in Section 3.2) independently, and together. This case is chosen only to illustrate the benefits of using all proposed methods in tandem.

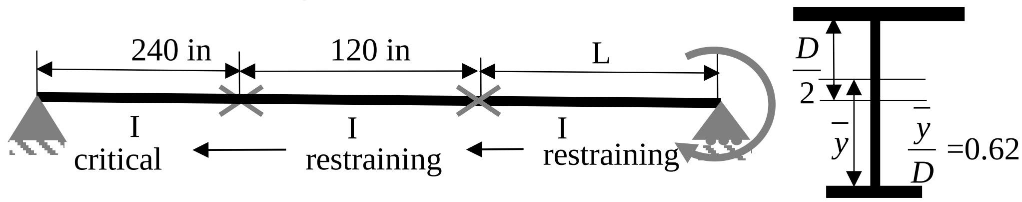

Figure 10: Loading for singly-symmetric case with smaller flange in compression

Figure 11 shows the comparison between the N&T method and the accuracy of each of the proposed methods with respect to the true capacities (FE results from ABAQUS).

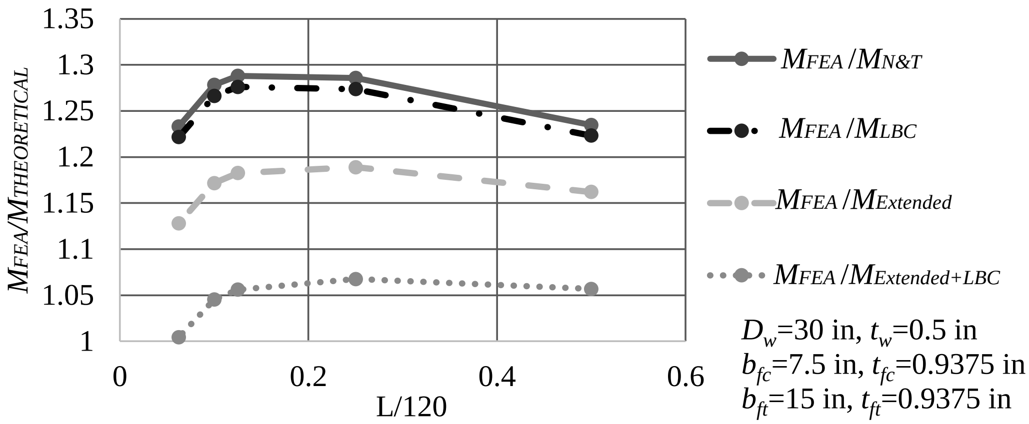

Figure 11: Comparison of various methods in the LTB prediction for singly-symmetric cross-sections

As seen in Figure 11, combining the two methods predicts the best results. Using the methods independently also predict better results compared to the N&T method. The critical span is the left-most span, but the effect of the right-most span is observed to be considerable. The N&T method, and the LBC method do not take into account this effect of a segment that is not immediately adjacent to the critical segment. This is accounted in the extended method, and hence, performs better than the LBC in this case. Combining the two methods give the best results as it captures both the effect of high moment gradients on unsymmetric boundary conditions and the restraining effects of spans more than two segments away from the critical span.

4. Illustrative examples using various methods

This Section provides sample calculations for the various methods referred to in this paper.

4.1 Nethercot and Trahair's method (Nethercot and Trahair 1976)

The steps to Nethercot and Trahair's method are as follows:

1. Calculate the buckling load using Equation (3) for each span assuming effective length factor (*K*) as unity and appropriate moment modification factors (C_b).
2. Identify the critical segment as the one with the smallest buckling load.
3. Calculate the stiffness ratios as shown in Equation (24).
4. Calculate the effective length factor (*K*) from the alignment chart for braced columns.
5. Calculate the buckling capacity using Equation (2)

$$\alpha_M = \frac{2EI_Y}{L_M} \Rightarrow \begin{array}{l} \alpha_{RL} = \dfrac{nEI_Y}{L_{RL}}\left(1 - \dfrac{P_M}{P_{RL}}\right) \Rightarrow G_L = \dfrac{\alpha_M}{\alpha_{RL}} \\ \alpha_{RR} = \dfrac{nEI_Y}{L_{RR}}\left(1 - \dfrac{P_M}{P_{RR}}\right) \Rightarrow G_R = \dfrac{\alpha_M}{\alpha_{RR}} \end{array} \tag{24}$$

where, *n* is two for far end having equal restraint as near end, *n* is three for far end hinged and *n* is four for far end fixed. The problem in Figure 2 is solved below using this method. The spans from left to right are numbered one through five.
The LTB load of each unbraced span calculated using Equation (2) are
$P_1 = P_5 = 1.75M$, $P_2 = P_3 = P_4 = M$
Assign segment three as critical, with spans 2 and 4 designated as the restraining segments.

$$\alpha_M = \frac{2EI_Y}{L_M} \Rightarrow \begin{array}{l} \alpha_{RL} = \dfrac{2EI_Y}{L_{RL}}\left(1 - \dfrac{P_3}{P_4}\right) = 0 \Rightarrow G_L = \dfrac{\alpha_M}{\alpha_{RL}} = \infty \\ \alpha_{RR} = \dfrac{2EI_Y}{L_{RR}}\left(1 - \dfrac{P_3}{P_4}\right) = 0 \Rightarrow G_R = \dfrac{\alpha_M}{\alpha_{RL}} = \infty \end{array}$$

From column charts, the effective length factor *K*=1.
Using Equation (3), $\boldsymbol{M_{cr} = M}$ **= 6747 in-kips** for cross-section with dimensions, D_w = 30 in, t_w = 0.5 in, b_f = 7.5 in, t_f =0.9375 in. Critical buckling load given by ABAQUS is **7391 in-kips.**

4.2 Extended method
This method refers to the modified procedure wherein the restraining effect of spans adjacent to the immediate restraining segment is also considered. The steps are as follows:
1. Identify the critical segment just as in the Nethercot and Trahair method.
2. Identify the restraining segments and calculate their buckling capacities using Nethercot and Trahair's method.
3. Calculate the stiffness ratios using the new buckling capacity of restraining segments from Step 2.
4. Calculate the effective length factor and buckling capacity of the critical segment.

The problem in Figure 2 solved above is again solved here using the extended method.
Similar to the Nethercot and Trahair method, Assign segment three as critical, with spans two and four designated as the restraining segments.
Calculate the buckling capacity of span two using Nethercot and Trahair's method:
Restraining segment for span two is span one, while span three is assumed to give no restraint as its buckling capacity is smaller than that of span two.

$$1)\alpha_m = \frac{2EI}{L},\ 2)\alpha_{r12} = \frac{2EI}{L}\left(1 - \frac{M_2}{M_1}\right) = \frac{2EI}{L}\left(1 - \frac{M}{1.75M}\right) = \frac{1.5EI}{1.75L},\ 3)G_{12} = \frac{\alpha_m}{\alpha_{r12}} = 1.55$$

$$4)\alpha_{r32} = \frac{2EI}{L}\left(1 - \frac{M_2}{M_3}\right) = 0,\ 5)G_{32} = \frac{\alpha_m}{\alpha_{r12}} = \infty$$

From the braced column alignment chart, $K_2 = 0.91$

For section with dimensions D_w = 30 in, t_w = 0.5 in, b_f = 7.5 in, t_f = 0.9375 in: using Equation (3), buckling capacity of span two is M_2 =7839 in-kips.

Calculate the buckling capacity of span four:

By symmetry, span four has the same capacity as span two, $M_4 = M_2$ =7839 in-kips

Calculate the buckling capacity of span three:

$$\alpha_m = \frac{2EI}{L},\ \alpha_{r23} = \frac{2EI}{L}\left(1 - \frac{M_3}{M_2}\right) = \frac{2EI}{L}\left(1 - \frac{6747}{7839}\right),\ G_{23} = \frac{\alpha_m}{\alpha_{r23}} = 7.18$$

$$\alpha_m = \frac{2EI}{L},\ \alpha_{r43} = \frac{2EI}{L}\left(1 - \frac{M_3}{M_4}\right) = \frac{2EI}{L}\left(1 - \frac{6747}{7839}\right), G_{43} = \frac{\alpha_m}{\alpha_{r12}} = 7.18$$

From the braced column alignment chart, K_3 =0.95

Using Equation (3), critical buckling load is $\boldsymbol{M_3}$ **=7318 in-kips**.

Critical buckling load given by ABAQUS is **7391 in-kips**.

4.3 Loading and boundary condition (LBC) interaction method

The steps to take into account loading and boundary condition interaction are as follows:

1. Calculate the stiffness ratios of critical member as suggested in Nethercot and Trahair (1976).
2. Calculate distribution factor (DF) using Equation (11).
3. Calculate β using Equation(12), select a and b based on whether DF < 1 or DF > 1.
4. Calculate the effective length factor using Equation (15).

To solve the problem in Figure 5(b) for L = 60in for cross section with dimensions D_w = 30 in, t_w = 0.5 in, b_f = 7.5 in, t_f =0.9375 in: spans from left to right are numbered one and two.

From Nethercot and Trahair's method, span two is critical and $G_{12} = 0.169$, $G_{right} = \infty$

$\boldsymbol{K_{N\&T}}$ **= 0.76.**

$$DF = \frac{M_{far}}{M_{near}} = \frac{0.75M_{right} + 0.25M_{left}}{0.25M_{right} + 0.75M_{left}} = 2 > 1$$

$$\beta_{12} = \frac{DF}{2}G_{12}(N\,\&T) + \frac{DF}{10} = 0.3691 \qquad \beta_{hinged} = G_{hinged} = \infty \quad \Rightarrow \boldsymbol{K_{new}} = \frac{3\beta_A\beta_B + 1.4(\beta_A + \beta_B) + 0.64}{3\beta_A\beta_B + 2(\beta_A + \beta_B) + 1.28} = \mathbf{0.81}$$

The effective length factor predicted by SABRE2, $\boldsymbol{K_{SABRE2}}$ **= 0.81**.

4.4 The iteration method

The iteration method follows as:

1. Calculate the buckling load (P_0) for each span with K=1 to identify the critical segment and estimate the stiffness ratios.
2. Calculate the critical buckling load (P_1) following Nethercot and Trahair's method.
3. Use the buckling load P_1 to re-identify the critical segment and estimate the stiffness ratios if P_1 is larger than the buckling load of its restraining segments.
4. Calculate the final critical buckling load P_2 following Nethercot and Trahair's method for the correct critical segment.

To solve the problem in Figure 8 with L=120 in. for cross section with dimensions D_w = 30 in, t_w = 0.5 in, b_f= 7.5 in, t_f=0.9375 in: spans from left to right are numbered one to three.

Calculate the buckling load for each span:

P_{01} = 116407 in-kips, P_{02} = 9671 in-kips, P_{03} = 11807 in-kips

Span 2 is identified as the critical segment, and the restraining segments are spans one and three.

Buckling loads are calculated using Nethercot and Trahair:
P_{11} = 116407 in-kips, P_{12} = **14774 in-kips**, P_{13} = 11807 in-kips
The critical segment is now identified as span three, and the restraining member is span two.
The buckling loads are once again calculated using the Nethercot and Trahair method:
P_{21} = 116407 in-kips, P_{22} = 14774 in-kips, $\boldsymbol{P_{23}}$ **= 12498 in-kips**
There is no further change in the identification of the critical member, and there is no significant difference (like 1.5 times) in P_{13} to P_{23}. Hence, further iteration is not required.
The buckling load predicted by SABRE2 is **12636 in-kips**. Nethercot and Trahair's method would predict a capacity equal to P_{12}=**14774 in-kips**.

4.5 Extended method + LBC method
This method is applicable for the case where the end span is critical. Proceed in the steps of the extended method with the exception of modifying the stiffness ratios (G) to β as directed in the LBC method.
To solve the problem in
Figure 10 for L=30in for singly-symmetric section with cross-section dimensions D_w=30 in, t_w=0.5 in, b_{fc}=7.5 in, t_{fc}=0.9375 in, b_{ft}=15in, t_{ft}=0.9375 in. Spans from left to right are numbered one to three.
Buckling capacity of each span using Equation(21):
P_1 = 22692 in-kips, P_2 = 30894 in-kips, P_3 = 356748 in-kips
Critical segment is span one and the restraining segment is span two
Calculating the buckling capacity of span two:
Restraining segment for span two is span three, while span one is assumed to give no restraining effect because its buckling capacity is lower than that of span two.
From the Nethercot and Trahair method $G_{12} = \infty$, G_{32} = 0.18
Distribution factor (DF) of span two = 0.82
Calculate β using Equation(12) as $DF < 1$:
$\beta_{12} = \infty$, β_{32} = 0.11
Effective length factor (K) using Equation(15), K = 0.74
Buckling capacity of span 2 using equation(21) is P_2 = 53222 in-kips
Calculate the buckling capacity of span 1:
First, calculate the restraining effect of span 2 using the new buckling load (P_2)
Stiffness ratios calculated as directed in Nethercot and Trahair, G_{21} = 0.87, $G_{hinge} = \infty$
Distribution factor (DF) of span one = 0.33
Calculate β using Equation(12) as $DF < 1$:
$\beta_{hinge} = \infty$, β_{21} = 0.4
Effective length factor (K) using Equation(15), K = 0.81,
Critical buckling capacity calculated using Equation(21) is $\boldsymbol{P_1}$ **= 30649 in-kips**
Critical buckling capacity predicted by SABRE2 is $\boldsymbol{P_{FEA}}$ **= 32718 in-kips**
Critical buckling capacity predicted by N&T method is $\boldsymbol{P_{N\&T}}$ **= 25446 in-kips**.

5. Conclusions
Design engineers often look for simple hand-based techniques, rather than advanced analysis tools to estimate the LTB capacities of beams with multiple unbraced spans. Some of the drawbacks of the existing Nethercot and Trahair method are discussed and various methods are proposed in this paper to reduce the error in effective length prediction. While it is obvious why unconservative predictions using the N&T method is undesirable, this paper has also shown how

one may incorrectly formulate the design LTB curve while using conservative predictions from the N&T method. The key findings in the paper are summarized below:

1. The methods proposed capture the restraining effects of spans beyond the immediate adjacent segment. It is shown that although the N&T method is conservative compared to the proposed method (extended method) in such cases.
2. A modified procedure is proposed in this paper when the beam is subjected to a linear bending moment, and the end segment is critical. It is shown that the N&T method is overly unconservative when the maximum moment is at the pin support, and overly conservative when the maximum moment is at the brace location. The proposed procedure greatly reduces the error in these predictions. This is also validated for beams subjected to reverse curvature.
3. The paper presents potential conditions, where the critical segment may be wrongly identified, resulting in larger estimates of buckling capacities.
4. The paper also discusses conditions where the restraining effect of adjacent segments is grossly over-estimated leading to extremely unconservative solutions.
5. Finally, the paper has verified all the above procedures for singly-symmetric cross-sections, while also showing that where applicable, all the above modifications may be used simultaneously for the best results.

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